Syracuse Algebra Seminar
October 2021

non-unit
non-zero

Background on Matrix Factorizations.

Def. Let S be a regular local ring and $f \in S$.
A matrix factorization of f is a pair of square matrices (φ, ψ) with entries in S s.t.

$$\varphi\psi = f \cdot I_n = \psi\varphi.$$

Ex Let $f = x^3 + y^4 \in S = K[[x,y]]$

$$\underbrace{\begin{bmatrix} x & -y^2 \\ y^2 & x^2 \end{bmatrix}}_{\varphi} \cdot \underbrace{\begin{bmatrix} x^2 & y^2 \\ -y^2 & x \end{bmatrix}}_{\psi} = \begin{bmatrix} x^3+y^4 & 0 \\ 0 & x^3+y^4 \end{bmatrix} = \psi\varphi$$

So, (φ, ψ) is a 2×2 matrix factorization of $x^3 + y^4$

Let $R = S/(f)$ be the hypersurface ring defined by f.

MFs and R-modules.

Let (φ, ψ) be a MF of f of size n.

Since $\varphi\psi = f \cdot I_n$,

$$f(S^n) = \varphi\psi(S^n) = \varphi(\psi S^n) \subseteq \operatorname{Im}\varphi$$

$\Rightarrow f$ kills $\operatorname{cok}\varphi = S^n/$

$\Rightarrow$ f kills $\operatorname{Cok}\varphi = S^n / \operatorname{Im}\varphi$

So, $\operatorname{Cok}\varphi$ is naturally an $R = S/(f)$ - module

(same for $\operatorname{Cok}\psi$)

Since $f \neq 0$ is a nzd, φ and ψ are injective:

$$0 \longrightarrow S^n \xrightarrow{\varphi} S^n \longrightarrow \operatorname{Cok}\varphi \longrightarrow 0$$

Recall/Def. $R = S/(f)$

(1) A fin gen R-module M is called maximal Cohen-Macaulay (MCM) if $\operatorname{depth} M = \dim R$.

(2) Auslander-Buchsbaum: An R-module M is MCM over R if and only if $\operatorname{pd}_S M = 1$.

For a MF (φ, ψ)

- $\operatorname{Cok}\varphi$ is an R-module
- $\operatorname{pd}_S(\operatorname{Cok}\varphi) = 1$

} So, $\operatorname{Cok}\varphi$ is an MCM R-module (same for $\operatorname{Cok}\psi$)

Conversely. Let M be any MCM R-module.

By above, $\operatorname{pd}_S M = 1$, so $\exists$

M is torsion
$f \cdot M = 0$
So, $\operatorname{rank}_S M = 0$

$$\begin{tikzcd}
 & & S^n \arrow[dll, dashed, "\exists!\psi"'] \arrow[d, "f"] \arrow[drr, "0"] \arrow[dl, phantom, "\circlearrowleft"] & & \\
0 \arrow[r] & S^{\cancel{m}\,n} \arrow[r, "\varphi"] & S^n \arrow[rr] & & M \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \varphi\psi = f \cdot I_n \Longrightarrow \psi\varphi = f \cdot I_n$

$\Rightarrow (\varphi, \psi)$ is a MF of f s.t. $\operatorname{Cok}\varphi = M$.

Thm (Eisenbud) The correspondence

$$(\varphi, \psi) \longleftrightarrow \operatorname{Cok}\varphi$$

forms a bijection between reduced MFs and MCM R-modules w/ no free summands.

- reduced means the entries of φ and ψ live in the max'l ideal of S.
- free summand means $M \cong R \oplus M'$ for some M'.

$$* \quad \text{MFs} \longleftrightarrow \text{MCMs} \quad *$$

Motivating Theorem (BGS-Knörrer)

Let $S = k[\![x, y, z_2, \ldots, z_r]\!]$ for $k = \bar{k}$, char $k = 0$. TFAE

i) R is a simple hypersurface singularity

ii) R has finite Cohen-Macaulay type

iii) $R \cong S/(g + z_2^2 + z_3^2 + \cdots + z_r^2)$ where g is

$(A_n)\quad x^2 + y^{n+1} \qquad n \geq 1$

$(D_n)\quad x^2y + y^{n-1} \qquad n \geq 4$

$(E_6)\quad x^3 + y^4$

$(E_7)\quad x^3 + xy^3$

$(E_8)\quad x^3 + y^5$

- R is a simple hypersurface singularity if the set of proper ideals of S

$$c(f) = \{ I \subsetneq S \mid f \in I^2 \}$$

is finite.

- R has finite CM type if there are, up to iso, only finitely many indecomposable MCM R-modules.

The Thm
- Classifies hyp rings of fCMt — Rep. Theory of local rings
- Connects w/ simple singularities — Deformations in Alg. Geom.

Notice: R has finite CM type $\iff$ there are only finitly many iso classes of MFs of f.

Key Ingredient — to contribution made by Knörrer

Skew Group Algebra. char $k \neq 2$

- $R = S/(f)$

- $R^\# = S[\![z]\!] / (f + z^2)$ ← double branched cover of R
- $\exists$ an automorphism $\sigma : R^\# \to R^\#$

  $\sigma(s) = s$ for all $s \in S$

  $\sigma(z) = -z$

Can form the skew group algebra $R^\#[\sigma]$

- $R^\#[\sigma] =$ formal sums $a + b\sigma$, $a, b \in R^\#$
- multiplication given by: $a, b \in R^\#$

$$(a \cdot \sigma^i) \cdot (b \cdot \sigma^j) = a\,\sigma^i(b) \cdot \sigma^{i+j}$$

Knörrer:

(1) $MF(f) \simeq MCM(R^\#[\sigma]) = \left\{ \begin{array}{l} R^\#[\sigma]\text{-modules} \\ \text{fin gen free over } S \end{array} \right\}$

(2) R has finite CM type $\Longleftrightarrow$ $R^\#$ has finite CM type

$\Updownarrow$ $\Updownarrow$

f has finite MF type $\Longleftrightarrow$ $f + z^2$ has finite MF type

d-fold Matrix Factorizations.

(or d-fold)

Def. $f \in S$, $d \geq 2$. A matrix factorization of f with d factors is: $(\varphi_1, \varphi_2, \ldots, \varphi_d)$ $n \times n$ matrices w/ entries in S s.t.

$$\varphi_1 \varphi_2 \cdots \varphi_d = f \cdot I_n$$

Notice: $\varphi_i \varphi_{i+1} \cdots \varphi_d \varphi_1 \cdots \varphi_{i-1} = f \cdot I_n$ for all i.

Ex $f = xyz \in K[[x,y,z]]$. Then

(x, y, z) is a 3-fold MF of $f = xyz$

size 1×1

Ex $f = x^3 + y^4 \in K[[x,y]]$. Assume $\omega \in K$: $\omega^3 = 1$, a primitive 3^{rd} root of 1.

$$\left(\underbrace{\begin{pmatrix} y^2 & 0 & x \\ x & y & 0 \\ 0 & x & y \end{pmatrix}}_{\varphi_1}, \underbrace{\begin{pmatrix} y & 0 & \omega x \\ \omega x & y & 0 \\ 0 & \omega x & y^2 \end{pmatrix}}_{\varphi_2}, \underbrace{\begin{pmatrix} y & 0 & \omega^2 x \\ \omega^2 x & y^2 & 0 \\ 0 & \omega^2 x & y \end{pmatrix}}_{\varphi_3} \right)$$

size 3×3

is a 3-fold MF of $f = x^3 + y^4$.

Thm (–) S is complete, $K = \bar{K}$, char $K \nmid d$.

Let $d \geq 2$, $f \in S$, $\omega \in S$ a prim d^{th} root of 1.

- $R = S/(f)$
- $R^\# = S[[z]]/(f + z^d)$ $\leftarrow$ d-fold branched cover of R
- $\exists$ $\sigma : R^\# \longrightarrow R^\#$

  $\sigma(s) = s$

  $\sigma(z) = \omega z$

Form $R^{\#}[\sigma]$ as before. Then,

$$\underset{\substack{\text{category of } d\text{-fold MFs} \\ \text{of } f}}{MF^{d}(f)} \simeq MCM(R^{\#}[\sigma]) = \left\{ \begin{array}{c} R^{\#}[\sigma]\text{-modules} \\ \text{f.g. free over } S \end{array} \right\}$$

Idea behind the equivalence:

$N \in MCM(R^{\#}[\sigma]) \rightsquigarrow N \in MCM(R^{\#})$, N f.g. free over S.

Let $\varphi : N \to N$ be mult by z on N.

Pick S-basis for N and write φ $\underbrace{\text{as a matrix w/ entries in } S}_{\text{no } z\text{'s}}$

Then, $\varphi^{d} = \text{mult by } z^{d} = -f \cdot I_n$

$\Rightarrow$ get a $\underset{\text{with } d\text{-factors}}{\text{matrix factorization}}$ of f $\approx (\varphi, \varphi, \ldots, \varphi)$.

$$\begin{tikzcd}
MCM(R^{\#}) \arrow[d, shift right] \arrow[drr, bend left, "\flat"] & & \\
MCM(R^{\#}[\sigma]) \arrow[u, shift right] \arrow[rr, leftrightarrow, "\sim"] & & MF^{d}(f) \arrow[ull, bend right, "\#"]
\end{tikzcd}$$

$\#$ and $\flat$ do not form an equivalence but:

Let $N \in MCM(R^{\#})$ and $X \in MF^{d}(f)$.

$$N^{\flat\#} \cong \bigoplus_{i=0}^{d-1} (\sigma^i)^* N \qquad\qquad X^{\#\flat} \cong \bigoplus_{i=0}^{d-1} T^i(X)$$

where
- $(\sigma^i)^* N$ is the module obtained by restricting scalars along $\sigma^i : R^\# \to R^\#$
- $T^i(\varphi_1, \varphi_2, \ldots, \varphi_d) = (\varphi_i, \varphi_{i+1}, \ldots, \varphi_d, \varphi_1, \ldots, \varphi_{i-1})$

Say f has finite d-MF type if there are, up to iso, finitely many indecomposable d-fold MFs of f.

Thm (Leuschke, –)

f has finite d-MF type iff $R^\# = S[\![z]\!]/(f+z^d)$ has finite CM type.

Cor. Let $S = K[\![y, z_2, \ldots, z_r]\!]$, $K = \bar{K}$, char $K = 0$, $d > 2$.
f has finite dMF type iff f and d are one of:

(A_1)	$y^2 + z_2^2 + \cdots + z_r^2$	for any $d > 2$
(A_2)	$y^3 +$ " "	for $d = 3, 4, 5$
(A_3)	$y^4 +$ " "	for $d = 3$
(A_4)	$y^5 +$ " "	for $d = 3$.