Branched covers and matrix factorizations

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Setup

Let S be a regular local ring and $f \in S$ a non-zero non-unit.

- The **hypersurface ring** defined by f: $R = S/(f)$
- The **double branched cover** of R: $R^\sharp = S[\![z]\!]/(f + z^2)$

Motivating Question:

How does the representation theory of R compare to the representation theory of $R^\sharp$?

We will also assume that S is complete over an algebraically closed field of characteristic $\neq 2$.

MCM modules

- A finitely generated module M over a local ring A is called **maximal Cohen-Macaulay** (MCM) if

$$\operatorname{depth}_A(M) = \dim A.$$

For this talk:
An R-module M is MCM $\iff$ M has projective dimension 1 over S.
(resp. an $R^\sharp$-module) (resp. over $S[\![z]\!]$)

- A local ring A is said to have **finite Cohen-Macaulay representation type** if there are, up to isomorphism, only finitely many indecomposable MCM A-modules.

Theorem (Knörrer)

$$R \text{ has finite CM type} \iff R^\sharp \text{ has finite CM type.}$$

Main Tool

A **matrix factorization** of f is a pair of $n \times n$ matrices (φ, ψ) with entries in S such that

$$\varphi\psi = f \cdot I_n = \psi\varphi$$

Example

Let $f = x^3 + y^4$

$$\begin{pmatrix} x & -y^2 \\ y^2 & x^2 \end{pmatrix} \begin{pmatrix} x^2 & y^2 \\ -y^2 & x \end{pmatrix} = \begin{pmatrix} x^3 + y^4 & 0 \\ 0 & x^3 + y^4 \end{pmatrix}$$

MFs and MCM modules

Given a matrix factorization (φ, ψ) of f, both $\operatorname{cok}\varphi$ and $\operatorname{cok}\psi$ are MCM R-modules.

$$0 \to S^n \xrightarrow{\varphi} S^n \to \operatorname{cok}\varphi \to 0$$

The converse also holds and we have:

Theorem (Eisenbud)
The correspondence

$$(\varphi, \psi) \longleftrightarrow \operatorname{cok}\varphi$$

defines a bijection between **reduced** matrix factorizations of f and *MCM* R-modules with no free summands.

Knörrer's Theorem

The **skew group algebra** $R^\sharp[\sigma]$:

- $R = S/(f)$
- $R^\sharp = S[\![z]\!]/(f + z^2)$
- $\sigma \in \mathrm{Aut}(R^\sharp)$ which fixes S, $\sigma(z) = -z$

$R^\sharp[\sigma]$ has elements of the form $a + b \cdot \sigma$ for $a, b \in R^\sharp$ with multiplication:

$$(a \cdot \sigma^i) \cdot (b \cdot \sigma^j) = a\sigma^i(b) \cdot \sigma^{i+j}$$

Theorem (Knörrer 1987)

1. There is an equivalence of categories

$$\mathrm{MF}(f) \approx MCM(R^\sharp[\sigma]) = \left\{ R^\sharp[\sigma]\text{-modules which are MCM over } R^\sharp \right\}$$

2. R has finite CM type if and only if $R^\sharp$ has finite CM type

Simple Singularities

$R = S/(f)$ is called a **simple hypersurface singularity** if there exists finitely many proper ideals $I \subsetneq S$ such that $f \in I^2$.

Theorem (Knörrer, Buchweitz-Greuel-Schreyer 1987)
Let $S = \mathbf{k}[\![x, y, z_2, \dots, z_r]\!]$ where $\mathbf{k} = \overline{\mathbf{k}}$ and char $\mathbf{k} = 0$. Then the following are equivalent.

1. R is a simple hypersurface singularity
2. R has finite Cohen-Macaulay type
3. $R \cong \mathbf{k}[\![x, y, z_2, \dots, z_r]\!]/(g)$ where g is one of the following

(A_n) $x^2 + y^{n+1} + z_2^2 + \cdots + z_r^2 \quad n \geq 1,$
(D_n) $x^2y + y^{n-1} + z_2^2 + \cdots + z_r^2 \quad n \geq 4,$
(E_6) $x^3 + y^4 + z_2^2 + \cdots + z_r^2$
(E_7) $x^3 + xy^3 + z_2^2 + \cdots + z_r^2$
(E_8) $x^3 + y^5 + z_2^2 + \cdots + z_r^2$

Matrix factorizations with more factors

Matrix factorizations with $d \geq 2$ factors

Definition
A matrix factorization of f with d factors is:

An ordered tuple of $n \times n$ matrices $(\varphi_1, \varphi_2, \ldots, \varphi_d)$ with entries in S such that

$$\varphi_1 \varphi_2 \cdots \varphi_d = f \cdot I_n$$

$\big($Notice that $\varphi_i \varphi_{i+1} \cdots \varphi_d \varphi_1 \cdots \varphi_{i-1} = f \cdot I_n$ for all $i\big)$

$\mathrm{MF}^d(f) =$ category of d-fold factorizations of f

Examples

- $(x, y, z) \in \mathrm{MF}^3(xyz)$
- Let $f = x^3 + y^4$ and $\omega^3 = 1$.

$$\left(\begin{pmatrix} y^2 & 0 & x \\ x & y & 0 \\ 0 & x & y \end{pmatrix}, \begin{pmatrix} y & 0 & \omega x \\ \omega x & y & 0 \\ 0 & \omega x & y^2 \end{pmatrix}, \begin{pmatrix} y & 0 & \omega^2 x \\ \omega^2 x & y^2 & 0 \\ 0 & \omega^2 x & y \end{pmatrix} \right) \in \mathrm{MF}^3(f)$$

Theorem (T.)
Let $d \geq 2$ and assume char k does not divide d.

- $R = S/(f)$
- $R^\sharp = S[\![z]\!]/(f + z^d)$ — *the* d***-fold branched cover*** *of* R
- $\sigma : R^\sharp \to R^\sharp$ *fixes* S *and* $\sigma(z) = \omega z \qquad (\omega^d = 1)$

There is an equivalence of categories

$$MF^d(f) \approx MCM(R^\sharp[\sigma]) = \left\{ R^\sharp[\sigma]\text{-modules which are MCM over } R^\sharp \right\}$$

Idea behind the equivalence
$N \in \mathrm{MCM}(R^\sharp[\sigma]) \implies N$ is an $R^\sharp$-module and a free S-module. Let $\varphi : N \to N$ be multiplication by z. Then

$$\varphi^d = -f \cdot 1_N$$

This gives us a d-MF of the form $\approx (\varphi, \varphi, \ldots, \varphi) \in \mathrm{MF}^d(f)$

MCM($R^\sharp$) and MF$^d(f)$

- Notice that this construction applies to $N \in \mathrm{MCM}(R^\sharp)$. Get a functor

$$\begin{aligned} \mathrm{MCM}(R^\sharp) &\xrightarrow{\quad\flat\quad} \mathrm{MF}^d(f) \\ N &\longmapsto N^\flat \approx (\varphi, \varphi, \dots, \varphi) \end{aligned}$$

- We also have a functor

$$\begin{aligned} \mathrm{MCM}(R^\sharp) &\xleftarrow{\quad\sharp\quad} \mathrm{MF}^d(f) \\ X^\sharp &\longleftarrow\!\shortmid X \end{aligned}$$

* These do not form an equivalence.

Proposition

Let N be an MCM $R^\sharp$-module and $X \in \mathrm{MF}^d(f)$. Then

$$N^{\flat\sharp} \cong \bigoplus_{i=0}^{d-1} (\sigma^i)^* N \quad \text{and} \quad X^{\sharp\flat} \cong \bigoplus_{i=0}^{d-1} T^i(X)$$

where

- $(\sigma^i)^* N$ is the module obtained by restriction scalars along $\sigma^i : R^\sharp \to R^\sharp$
- $T^i(\varphi_1, \varphi_2, \ldots, \varphi_d) = (\varphi_i, \varphi_{i+1}, \ldots, \varphi_d, \varphi_1, \ldots, \varphi_{i-1})$

Finite type

Say that f has **finite d-MF type** if there are, up to isomorphism, only finitely many indecomposable matrix factorizations f with d factors.

Theorem (Leuschke, T.)
f has finite d-MF type if and only if the d-fold branched cover $R^\sharp = S[\![z]\!]/(f + z^d)$ has finite CM type.

Corollary
Let $S = \mathbf{k}[\![y, z_2, \ldots, z_r]\!]$ where $\mathbf{k} = \overline{\mathbf{k}}$, char $\mathbf{k} = 0$, and $d > 2$. Then f has finite d-MF type if and only if f and d are one of the following:

(A_1) $y^2 + z_2^2 + \cdots + z_r^2$ for any $d > 2$,

(A_2) $y^3 + z_2^2 + \cdots + z_r^2$ for $d = 3, 4, 5$

(A_3) $y^4 + z_2^2 + \cdots + z_r^2$ for $d = 3$

(A_4) $y^5 + z_2^2 + \cdots + z_r^2$ for $d = 3$

Thank you!