What is a matrix factorization?

Consider the polynomial $x^3 + y^4 \in \mathbb{C}[x,y]$.

$x^3 + y^4$ is irreducible in $\mathbb{C}[x,y]$ but ...

$$\underbrace{\begin{bmatrix} x & -y^2 \\ y^2 & x^2 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x^2 & y^2 \\ -y^2 & x \end{bmatrix}}_{B} = \begin{bmatrix} x^3+y^4 & 0 \\ 0 & x^3+y^4 \end{bmatrix}$$

it can be factored using, in this case, 2×2 matrices. Such a pair of matrices, (A,B) with $A \cdot B = (x^3+y^4)\cdot I_2$, is called a matrix factorization of x^3+y^4.

Why would we want to do such a thing?

0) Neat ✓

1) Each time we "factor" x^3+y^4 in this way we are specifying a module over the ring

$$R = \mathbb{C}[x,y] \big/ (x^3+y^4) \quad \Big\}\text{ "hypersurface ring"}$$

$$\underset{\text{matrix factorization}}{(A,B)} \longmapsto \underset{R\text{-module}}{\operatorname{Coker} A}$$

For instance, here is a <u>different</u> factorization of x^3+y^4

$$(A', B') = \left(\begin{bmatrix} x & -y \\ y^3 & x^2 \end{bmatrix}, \begin{bmatrix} x^2 & y \\ -y^3 & x \end{bmatrix} \right)$$

Which gives a <u>different</u> R-module $\operatorname{Coker} A'$.

This relationship forms a powerful dictionary for translating questions about the ring R into simple Linear Algebra!

My work has been focused on a generalization. Namely: A matrix factorization of a polynomial f with $d \geq 2$ factors is a d-tuple of $n \times n$ matrices $(A_1, A_2, \ldots, A_d)$ such that

$$\boxed{A_1 A_2 \cdots A_d = f \cdot I_n \quad .}$$

<u>Ex</u> Returning to x^3+y^4 we have a factorization with 3 factors:

$$\left(\underset{A}{\begin{pmatrix} y & 0 & x \\ x & y^2 & 0 \\ 0 & x & y \end{pmatrix}}, \underset{B}{\begin{pmatrix} y^2 & 0 & \omega x \\ \omega x & y & 0 \\ 0 & \omega x & y \end{pmatrix}}, \underset{C}{\begin{pmatrix} y & & \omega^2 x \\ \omega^2 x & y & \\ & \omega^2 x & y^2 \end{pmatrix}} \right)$$

where $\omega \in \mathbb{C}$ is a primitive 3^{rd} root of unity.
Using the fact that $1+\omega+\omega^2=0$, you will find that

$$A \cdot B \cdot C = (x^3+y^4) \cdot I_3 .$$