Abstract

Let S be a regular local ring and f a non-zero non-invertible element of S. In this thesis, we study the notion of a matrix factorization of f with $d \geq 2$ factors, that is, we consider tuples of square matrices $(\varphi_1, \varphi_2, \ldots, \varphi_d)$, with entries in S, such that their product is f times an identity matrix of the appropriate size. These objects have been studied thoroughly in the case $d = 2$ and were originally introduced by Eisenbud in his study of free resolutions of modules over hypersurface rings. Many of the results given in this thesis are extensions of well-known results in the $d = 2$ case while others give new and unexpected properties which only arise when $d > 2$.

First we investigate the structure of the category of matrix factorizations with $d \geq 2$ factors in Chapter 2. We show that the stable category of d-fold matrix factorizations is naturally triangulated and we give an explicit formula for the relevant suspension functor. In Chapters 3 and 4 we give two different module-theoretic descriptions of this category, which turn out to be equivalent under mild assumptions, extending results of Solberg and Knörrer to the case of $d \geq 2$ factors.

The primary motivation for Chapter 4 is a theorem due to Knörrer which states that the category of 2-fold matrix factorizations of f has finite representation type if and only if the same is true of $f + z^2 \in S[\![z]\!]$, where z is an indeterminate. We consider an analogue of this statement in the case of the equation $f + z^d \in S[\![z]\!]$, $d \geq 2$. In particular, we show that there are, up to isomorphism, only finitely many indecomposable d-fold matrix factorizations of f if and only if the hypersurface ring defined by $f + z^d$ has finite Cohen-Macaulay representation type.

In Chapter 5, we provide a generalization of Eisenbud's fundamental theorem on the connection between matrix factorizations of f and maximal Cohen-Macaulay modules over

the hypersurface ring defined by f. Namely, we give a correspondence between d-fold matrix factorizations of f and sequences of $d-1$ surjective homomorphisms between the aforementioned modules.

Finally, Chapter 6 contains a formula for a tensor product of d-fold matrix factorizations in the sense of Yoshino as well as some criteria for decomposability of the construction.

Matrix Factorizations with More than Two Factors

by

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B.S., University of Rochester, 2016

M.S., Syracuse University, 2018

Dissertation

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

Syracuse University

May 2022

Acknowledgements

I would first like to thank my advisor, Graham Leuschke. Graham, I am incredibly grateful for the guidance and encouragement you have given me over the past six years. Your approach to mathematics and mentoring has made a permanent impression on me. Working with you has been intellectually stimulating and genuinely fun. I will miss our weekly meetings.

I would also like to thank the rest of my professors and mentors in the Mathematics Department. In particular, Professors Claudia Miller, Dan Zacharia, and Declan Quinn have been instrumental in my success at Syracuse. I am lucky to have been exposed to so many different areas of algebra. I also want to thank Professor Naomi Jochnowitz who saw potential in me and convinced me to pursue graduate school.

To my friends and classmates at Syracuse, thank you for your support and comradery. Especially Nathan, who has been stuck with me for six years. Thank you for putting up with me and for laughing at my jokes. You're a great friend and I'm glad we went through this together.

Thank you to my family: Guerras, Portillas, Donahues, and Tribones. I have received an immense amount of support from all of you. I am especially grateful for my parents who have supported me without question while I transitioned through life and finally wound up studying math.

Finally, I would like to thank Dalyana. Without you, none of this would be possible. Your unwavering support and belief in me has given me the strength to persevere through graduate school. It has been a privilege to be your partner, classmate, and colleague and I am incredibly excited for our next chapter.

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1 | Introduction

Let $(S, \mathfrak{n})$ be a regular local ring, that is, a commutative Noetherian ring that has a unique maximal ideal $\mathfrak{n}$ which is minimally generated by precisely $\dim S$ elements. The primary goal of the five chapters of this thesis is to study the notion of a d-fold matrix factorization of f where f is a fixed element of S and $d \geq 2$ is an arbitrary integer. In the case $d = 2$, these objects were introduced by Eisenbud in 1980 [Eis80] to study free resolutions of modules over hypersurface rings. Since then, the $d = 2$ case has been studied thoroughly in commutative algebra and related fields including Knot Theory and Physics. Many of the results given here are extensions of well-known results in the $d = 2$ case while others highlight new and unexpected properties that only arise when $d > 2$.

1.1 Summary of results

The first three chapters, after this introduction, are presented in order of increasing strength of assumptions needed on the regular ring S. The main results of Chapters 2 and 3 hold for an arbitrary regular local ring. Completeness of the ring S is needed in Chapter 3 in order to conclude that the Krull-Remak-Schmidt theorem holds in $\mathrm{MF}_S^d(f)$, the category of matrix factorizations with d factors, and throughout the entirety of Chapter 4. Furthermore, Chapter 4 requires that the residue field of S is algebraically closed of characteristic not dividing the fixed integer d. The majority of the results in the last chapter are stated in the case that S is a power series ring over a field which contains all the roots of $x^d \pm 1$ depending on the parity of d.

In Chapter 2 we investigate the category of matrix factorizations with d factors. We show that there is a suitable analogue for exact sequences in the additive category $\mathrm{MF}_S^d(f)$ with especially nice properties. Namely, there is an exact structure which has enough projectives, enough injectives, and the classes of injective and projective objects coincide. In other words, $\mathrm{MF}_S^d(f)$ is a Frobenius category. One consequence that we will discuss is that the induced stable category is naturally triangulated. We go on to describe, explicitly, the syzygy and cosyzygy operations in this category therefore also describing the suspension functor on the stable category.

Chapters 3 and 4 give module-theoretic descriptions of $\mathrm{MF}_S^d(f)$. Chapter 3 contains a direct extension of a result of Solberg [Sol89, Proposition 3.1] to the case of $d \geq 2$ factors while Chapter 4 extends results of Knörrer from [Knö87]. In particular, both Solberg and Knörrer identify the category of 2-fold matrix factorizations with a subcategory of modules over a non-commutative ring and we extend both of these results to the case of matrix factorizations with $d \geq 2$ factors. Chapter 4 also considers representation-theoretic questions about the category of d-fold matrix factorizations. The main result in this direction, which is joint with G. Leuschke, is Theorem 4.3.7 which can be viewed as an analogue of [Knö87, Corollary 2.8]. We show that the category of d-fold matrix factorizations is representation finite if and only if a certain hypersurface ring, called the d-fold branched cover, has finite Cohen-Macaulay type.

In Chapter 5, we generalize Eisenbud's fundamental result on the connection between matrix factorizations and maximal Cohen-Macaulay modules [Eis80, Corollary 6.3] to the case of factorizations with $d \geq 2$ factors. The main result of this chapter, which is an elaboration on the key idea identified by Hopkins in his thesis [Hop21, Theorem 3.14], shows that there is a correspondence between d-fold matrix factorizations of f and chains of surjections of length $d-1$ between maximal Cohen-Macaulay modules over the hypersurface ring defined by f.

Chapter 6 is dedicated to a d-fold version of a construction originally described by Knörrer in [Knö87]. Namely, we define a tensor product of d-fold matrix factorizations and investigate some of its basic properties using [Yos98] as a guide. The construction we give is based on

[BES17, Proposition 2.1] and [HUB91, Theorem 1.2].

Finally, we include a short appendix which contains technical details about idempotents in the category of d-fold matrix factorizations.

1.2 Matrix factorizations

In this section, we collect the main definitions and notations as well as some key results that we will use throughout. We also recall a fundamental theorem of Eisenbud, and a few of its corollaries, regarding matrix factorizations and maximal Cohen-Macaulay modules.

Definition 1.2.1. Let S be a regular local ring, f a non-zero non-unit in S, and $d \geq 2$ an integer. A *matrix factorization of f with d factors* is a d-tuple of homomorphisms between finitely generated free S-modules of the same rank, $(\varphi_1 : F_2 \to F_1, \varphi_2 : F_3 \to F_2, \ldots, \varphi_d : F_1 \to F_d)$, such that

$$\varphi_1\varphi_2\cdots\varphi_d = f \cdot 1_{F_1}.$$

Depending on the context, we may omit the free S-modules in the notation and simply write $(\varphi_1, \varphi_2, \ldots, \varphi_d)$. If the free S-modules $F_1, \ldots, F_d$ are of rank n, we say $(\varphi_1, \varphi_2, \ldots, \varphi_d)$ is a matrix factorization of size n.

It will be convenient to adopt the following notational conventions.

Notation 1.2.2. The letter d will always be an integer indicating the number of factors in a matrix factorization. When d is clear from context, all indices are taken modulo d unless otherwise specified. More specifically, let $i \neq j \in \mathbb{Z}_d = \{1, 2, \ldots, d\}$ and let $A_1, A_2, \ldots, A_d$ be symbols indexed over $\mathbb{Z}_d$. Let $\tilde{i}$ and $\tilde{j}$ be integer representatives of i, j within the range $0 < \tilde{i}, \tilde{j} \leq d$. The notation $A_iA_{i+1}\cdots A_j$ will be taken to mean

$$\begin{cases} A_iA_{i+1}\cdots A_{j-1}A_j & \text{if } \tilde{i} \leq \tilde{j} \\ A_iA_{i+1}\cdots A_dA_1\cdots A_{j-1}A_j & \text{if } \tilde{i} \geq \tilde{j} \end{cases}.$$

We follow a similar convention for indexing a decreasing list of symbols over $\mathbb{Z}_d$.

Definition 1.2.3. Let $(A, \mathfrak{m})$ be a local ring and $M \neq 0$ a finitely generated A-module. A sequence of elements $x_1, x_2, \ldots, x_c \in \mathfrak{m}$ is called an *M-regular sequence* if x_1 is a non-zerodivisor on M and, for each $i \geq 2$, x_i is a non-zerodivisor on $M/(x_1, x_2, \ldots, x_{i-1})M$. The well-defined constant, $\operatorname{depth}_A(M)$, which keeps track of the length of the longest M-regular sequence, is called the *depth* of M. A non-zero finitely generated A-module M is called *maximal Cohen-Macaulay* (MCM) if $\operatorname{depth}_A(M) = \dim A$, where $\dim A$ denotes the Krull dimension of M. The ring A is called *Cohen-Macaulay* if it is MCM as a module over itself.

Our first observation is that matrix factorizations of f with d factors encode MCM modules over the *hypersurface ring* $R = S/(f)$.

Lemma 1.2.4. *Let S be a regular local ring and f a non-zero non-unit in S. Let $(\varphi_1 : F_2 \to F_1, \varphi_2 : F_3 \to F_2, \ldots, \varphi_d : F_1 \to F_d)$ be a matrix factorization of f with $d \geq 2$ factors. For any $k \in \mathbb{Z}_d$,*

(i) $\varphi_k \varphi_{k+1} \cdots \varphi_{k-1} = f \cdot 1_{F_k}$, *and*

(ii) if $\operatorname{cok} \varphi_k \neq 0$, *then* $\operatorname{cok} \varphi_k$ *is an MCM R-module.*

Proof. (i) We proceed by induction on $d \geq 2$. For the case $d = 2$, we simply need to show that $\varphi\psi = f \cdot 1_F$ implies $\psi\varphi = f \cdot 1_G$. Suppose $(\varphi : G \to F, \psi : F \to G)$ is a matrix factorization with 2 factors, that is, suppose $\varphi\psi = f \cdot 1_F$. Since f is a non-zero element in the domain S, it follows that both φ and ψ are injective. Canceling φ on the left of the equation $\varphi\psi\varphi = f \cdot \varphi = \varphi \cdot f$, we find $\psi\varphi = f \cdot 1_G$.

Now, assume $d > 2$ and that the statement holds for matrix factorizations with fewer than d factors. Let $k \in \mathbb{Z}_d$ and notice that, by viewing the composition $\varphi_k\varphi_{k+1} : F_{k+2} \to F_k$ as a single homomorphism, the $(d-1)$-tuple

$$(\varphi_1, \varphi_2, \ldots, \varphi_{k-1}, \varphi_k\varphi_{k+1}, \varphi_{k+2}, \ldots, \varphi_d)$$

is a matrix factorization of f with $d-1$ factors. By induction, it follows that $\varphi_k\varphi_{k+1}\cdots\varphi_{k-1} = f\cdot 1_{F_k}$.

(ii) Let $k \in \mathbb{Z}_d$. By (i), we have that $\varphi_k\varphi_{k+1}\cdots\varphi_{k-1} = f\cdot 1_{F_k}$. In particular, $f\cdot \operatorname{cok}\varphi_k = 0$, that is, $\operatorname{cok}\varphi_k \neq 0$ is an R-module. Also, as in (i), the homomorphism φ_k is injective since $f \in S$ is non-zero. Thus, we have a short exact sequence

$$0 \longrightarrow F_{k+1} \xrightarrow{\varphi_k} F_k \longrightarrow \operatorname{cok}\varphi_k \longrightarrow 0,$$

which implies that $\operatorname{pd}_S(\operatorname{cok}\varphi_k) \leq 1$. By the Auslander-Buchsbaum formula, we have that

$$\operatorname{depth}(\operatorname{cok}\varphi_k) = \dim(S) - \operatorname{pd}_S(\operatorname{cok}\varphi_k) \geq \dim(S) - 1 = \dim(R).$$

That is, $\operatorname{cok}\varphi_k$ is an MCM R-module.

□

Definition 1.2.5. Let S be a regular local ring and f a non-zero non-unit in S. Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d)$ and $X' = (\varphi'_1 : F'_2 \to F'_1, \ldots, \varphi'_d : F'_1 \to F'_d)$ be matrix factorizations of f with $d \geq 2$ factors.

(i) A *morphism of matrix factorizations* between X and X' is a d-tuple of S-module homomorphisms, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, making each square of the following diagram commute:

$$\begin{array}{ccccccccc}
F_1 & \xrightarrow{\varphi_d} & F_d & \xrightarrow{\varphi_{d-1}} & \cdots & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\varphi_1} & F_1 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_d} & & & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} \\
F'_1 & \xrightarrow{\varphi'_d} & F'_d & \xrightarrow{\varphi'_{d-1}} & \cdots & \xrightarrow{\varphi'_2} & F'_2 & \xrightarrow{\varphi'_1} & F'_1.
\end{array}$$

Composition of morphisms is defined component-wise, that is, if $\alpha = (\alpha_1, \ldots, \alpha_d) : X \to X''$ and $\beta = (\beta_1, \ldots, \beta_d) : X' \to X$ are morphisms of matrix factorizations, then $\alpha \circ \beta = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_d\beta_d) : X' \to X''$. The matrix factorizations X and X' are isomorphic if there exists a morphism $\alpha = (\alpha_1, \ldots, \alpha_d) : X \to X'$ such that α_k is an isomorphism for each $k \in \mathbb{Z}_d$.

(ii) Let $\mathrm{MF}_S^d(f)$ denote the category of matrix factorizations of f with d factors with morphisms as above. The additive structure on $\mathrm{MF}_S^d(f)$ is given by the direct sum of X and X':

$$X \oplus X' := \left(\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_1' \end{pmatrix}, \begin{pmatrix} \varphi_2 & 0 \\ 0 & \varphi_2' \end{pmatrix}, \ldots, \begin{pmatrix} \varphi_d & 0 \\ 0 & \varphi_d' \end{pmatrix} \right).$$

(iii) We define functors $T^j : \mathrm{MF}_S^d(f) \to \mathrm{MF}_S^d(f)$, $j \in \mathbb{Z}_d$, given by

$$T^j(\varphi_1, \varphi_2, \ldots, \varphi_d) = (\varphi_{j+1}, \varphi_{j+2}, \ldots, \varphi_{j-1}, \varphi_j)$$

and

$$T^j(\alpha_1, \alpha_2, \ldots, \alpha_d) = (\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{j-1}, \alpha_j)$$

for any $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(X, X')$. We refer to $T = T^1$ as the *shift functor* on $\mathrm{MF}_S^d(f)$.

Definition 1.2.6. Let S be a regular local ring with maximal ideal $\mathfrak{n}$ and let f be a non-zero non-unit in S. Set $R = S/(f)$ and fix $d \geq 2$.

(i) Let $\mathrm{MCM}(R)$ denote the full subcategory of the category of finitely generated R-modules consisting of maximal Cohen-Macaulay R-modules.

(ii) An R-module M is *stable* if it has no direct summands isomorphic to R.

(iii) A matrix factorization $X = (\varphi_1, \varphi_2, \ldots, \varphi_d)$ is called *stable* if $\operatorname{cok} \varphi_k$ is a stable R-module for all $k \in \mathbb{Z}_d$.

(iv) A homomorphism between free S-modules $\varphi : G \to F$ is called *minimal* if $\operatorname{Im} \varphi \subseteq \mathfrak{n}F$.

(v) A matrix factorization $(\varphi_1, \varphi_2, \ldots, \varphi_d) \in \mathrm{MF}_S^d(f)$ is called *reduced* if φ_k is minimal for all $k \in \mathbb{Z}_d$.

(vi) For an R-module M, let $\mathrm{syz}_R^1(M)$ denote the *reduced* first syzygy of M, that is, $\mathrm{syz}_R^1(M)$ is the stable part of an arbitrary first syzygy of M over R.

(vii) A non-zero matrix factorization $X \in \mathrm{MF}_S^d(f)$ is *indecomposable* if $X \cong X' \oplus X''$ implies $X' = 0$ or $X'' = 0$.

The following theorem, due to Eisenbud, is the foundation for the theory of matrix factorizations with $d = 2$ factors.

Theorem 1.2.7 ([Eis80], Corollary 6.3)**.** *Let S be a regular local ring, $f \in S$ a non-zero non-unit, and $R = S/(f)$. The functor* $\operatorname{cok} : \mathrm{MF}_S^2(f) \to \mathrm{MCM}(R)$*, given by*

$$(\varphi, \psi) \in \mathrm{MF}_S^2(f) \longmapsto \operatorname{cok} \varphi \in \mathrm{MCM}(R),$$

induces a one-to-one correspondence between reduced matrix factorizations with 2 *factors and stable MCM R-modules.*

We set aside two consequences of Eisenbud's theorem that will be needed later.

Corollary 1.2.8 ([Eis80])**.** *Let S be a regular local ring, f a non-zero non-unit in S, and set $R = S/(f)$.*

(i) For any MCM R-module M, there exists a matrix factorization $(\varphi, \psi) \in \mathrm{MF}_S^2(f)$ such that φ is minimal and $\operatorname{cok} \varphi \cong M$.

(ii) A matrix factorization $(\varphi, \psi) \in \mathrm{MF}_S^2(f)$ is reduced if and only if it is stable. In this case, $\mathrm{syz}_R^1(\operatorname{cok} \varphi) \cong \operatorname{cok} \psi$ *and* $\mathrm{syz}_R^1(\operatorname{cok} \psi) \cong \operatorname{cok} \varphi$.

We will see in Section 2.2.2 that only one direction of (ii) holds when $d > 2$. Finally, we state another observation, also made by Eisenbud, that will help us identify matrix factorizations with more than two factors.

Lemma 1.2.9 ([Eis80], Corollary 5.4)**.** *Let S be a regular local ring and f a non-zero non-unit in S. Suppose $A : G \to F$ and $B : F \to G$ are homomorphisms of finitely generated free S-modules such that $AB = f \cdot 1_F$ and $BA = f \cdot 1_G$. Then* $\operatorname{rank} F = \operatorname{rank} G$.

Proof. Since $BA = f \cdot 1_G$, the map A is injective. Since $AB = f \cdot 1_F$, we have that $f \cdot \operatorname{cok} A = 0$, that is, $\operatorname{cok} A$ is a torsion S-module. After tensoring the short exact sequence

$$0 \longrightarrow G \xrightarrow{A} F \longrightarrow \operatorname{cok} A \longrightarrow 0$$

with the quotient field of S, we find that $\operatorname{rank}_S F = \operatorname{rank}_S G$. □

2 | Exact Structure

In this chapter we show that there is a natural choice of an exact structure on the category $\mathrm{MF}_S^d(f)$ which induces the structure of a triangulated category on the stable category $\underline{\mathrm{MF}}_S^d(f)$ defined below. We also give explicit formulas for the syzygy and cosyzygy operations and the cone of a morphism.

First, we will recall the axioms that define an exact category. The axioms and definitions below follow the presentation given in [Büh10] and we refer the reader to this paper for more information on exact categories.

2.1 Exact categories

Let $\mathcal{A}$ be an additive category. A pair of composable morphisms $A' \xrightarrow{i} A \xrightarrow{p} A''$ is called a *kernel-cokernel pair* if i is a kernel of p and p is a cokernel of i. Given a collection of kernel-cokernel pairs, $\mathcal{E}$, we call a morphism $i : A' \to A$ an *admissible monomorphism* if there exists a morphism $p : A \to A''$ such that $A' \xrightarrow{i} A \xrightarrow{p} A''$ is an element of $\mathcal{E}$. Dually, a morphism $p : A \to A''$ is an *admissible epimorphism* is there exists a morphism $i : A' \to A$ such that their composition is in $\mathcal{E}$. We will indicate admissible monomorphisms and admissible epimorphisms by the arrows $\rightarrowtail$ and $\twoheadrightarrow$ respectively.

An *exact structure* on $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which is closed under isomorphisms and such that the following axioms hold:

(E0) The identity morphism 1_X is an admissible monomorphism for all $X \in \mathcal{A}$.

(E0$^{\mathrm{op}}$) The identity morphism 1_X is an admissible epimorphism for all $X \in \mathcal{A}$.

(E1) Admissible monomorphisms are closed under composition.

(E1$^{\mathrm{op}}$) Admissible epimorphisms are closed under composition.

(E2) The push-out of an admissible monomorphism $X \rightarrowtail Y$ and an arbitrary morphism $X \to X'$ exists and induces an admissible monomorphism $X' \rightarrowtail Y'$ as in the diagram

$$\begin{array}{ccc} X & \rightarrowtail & Y \\ \downarrow & & \downarrow \\ X' & \rightarrowtail & Y'. \end{array}$$

(E2$^{\mathrm{op}}$) The pull-back of an admissible epimorphism $X' \twoheadrightarrow Y'$ and an arbitrary morphism $Y \to Y'$ exists and induces an admissible epimorphism $X \twoheadrightarrow Y$ as in the diagram

$$\begin{array}{ccc} X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \twoheadrightarrow & Y'. \end{array}$$

Given an additive category $\mathcal{A}$ and a class $\mathcal{E}$ satisfying these axioms, the pair $(\mathcal{A}, \mathcal{E})$ is called an *exact category*.

2.2 Short exact sequences of matrix factorizations

Let S be a regular local ring, $f \in S$ a non-zero non-unit, set $R = S/(f)$, and fix an integer $d \geq 2$. For the rest of this section, let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d), X' = (\varphi_1' : F_2' \to F_1', \ldots, \varphi_d' : F_1' \to F_d')$, and $X'' = (\varphi_1'' : F_2'' \to F_1'', \ldots, \varphi_d'' : F_1'' \to F_d'')$ be matrix factorizations in $\mathrm{MF}_S^d(f)$.

Definition 2.2.1. Suppose we have a pair of morphisms $\alpha = (\alpha_1, \ldots, \alpha_d) : X \to X''$ and $\beta = (\beta_1, \ldots, \beta_d) : X' \to X$ in $\mathrm{MF}_S^d(f)$. Then the composition

$$X' \xrightarrow{\beta} X \xrightarrow{\alpha} X''$$

is called a *short exact sequence of matrix factorizations* if the sequence

$$0 \longrightarrow F'_k \xrightarrow{\beta_k} F_k \xrightarrow{\alpha_k} F''_k \longrightarrow 0$$

is a short exact sequence of free S-modules for each $k \in \mathbb{Z}_d$.

Lemma 2.2.2. *A short exact sequence of matrix factorizations is a kernel-cokernel pair in* $\mathrm{MF}^d_S(f)$.

Proof. Let $X' \xrightarrow{\beta} X \xrightarrow{\alpha} X''$ be a short exact sequence of matrix factorizations. First we show that β is the kernel of α. By definition, we have that $\alpha\beta = 0$. Suppose $g : Y \to X$ is another morphism such that $\alpha g = 0$, where $Y = (\psi_1 : G_2 \to G_1, \dots, \psi_d : G_1 \to G_d) \in \mathrm{MF}^d_S(f)$. Let $k \in \mathbb{Z}_d$. We have the following diagram of free S-modules

$$\begin{array}{ccccccccc}
 & & & & G_k & & & & \\
 & & & \tilde{g}_k \swarrow & \downarrow g_k & \searrow 0 & & & \\
0 & \longrightarrow & F'_k & \xrightarrow{\beta_k} & F_k & \xrightarrow{\alpha_k} & F''_k & \longrightarrow & 0
\end{array}$$

where the bottom row is exact. Since β_k is the kernel of α_k, there exists a unique S-homomorphism $\tilde{g}_k : G_k \to F'_k$ such that $\beta_k \tilde{g}_k = g_k$. It suffices to show that $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_d) : Y \to X'$ is a morphism of matrix factorizations since each $\tilde{g}_k$, $k \in \mathbb{Z}_d$, is uniquely determined. That is, we need to show that the diagram

$$\begin{CD}
G_{k+1} @>{\psi_k}>> G_k \\
@V{\tilde{g}_{k+1}}VV @VV{\tilde{g}_k}V \\
F'_{k+1} @>{\varphi'_k}>> F'_k
\end{CD}$$

commutes for all $k \in \mathbb{Z}_d$. Note that $g_k \psi_k = \varphi_k g_{k+1}$ and $\varphi_k \beta_{k+1} = \beta_k \varphi'_k$ since g and β are morphisms in $\mathrm{MF}^d_S(f)$. Then $\beta_k \tilde{g}_k \psi_k = g_k \psi_k = \varphi_k g_{k+1} = \varphi_k \beta_{k+1} \tilde{g}_{k+1} = \beta_k \varphi'_k \tilde{g}_{k+1}$ and since β_k is injective, we can cancel it on the left to conclude that $\tilde{g}_k \psi_k = \varphi'_k \tilde{g}_{k+1}$ as desired. Hence,

$\tilde{g}$ is the unique morphism such that

$$\begin{aligned}\beta \circ \tilde{g} &= (\beta_1 \tilde{g}_1, \ldots, \beta_d \tilde{g}_d) \\ &= (g_1, \ldots, g_d) \\ &= g.\end{aligned}$$

This completes the proof that β is a kernel of α. The proof that α is a cokernel of β is similar. □

Let $\mathcal{E}_d$ denote the class of short exact sequences of matrix factorizations in $\mathrm{MF}_S^d(f)$. The first four axioms of an exact category are satisfied by the pair $(\mathrm{MF}_S^d(f), \mathcal{E}_d)$ directly from the definitions. The axioms (E2) and (E2$^{\mathrm{op}}$) also hold, which we will show below. Before we do, we need to know more about the form of the admissible morphisms in $(\mathrm{MF}_S^d(f), \mathcal{E}_d)$.

Lemma 2.2.3. *Let $\gamma = (\gamma_1, \ldots, \gamma_d) : X \to X''$ be a morphism of matrix factorizations.*

1. *γ is an admissible epimorphism if and only if the S-homomorphisms $\gamma_1, \ldots, \gamma_d$ are surjections.*

2. *γ is an admissible monomorphism if and only if the S-homomorphisms $\gamma_1, \ldots, \gamma_d$ are split injections.*

Proof. We prove only (2) as the proof of (1) is similar. Suppose γ is an admissible monomorphism. Then there exists an admissible epimorphism $\pi = (\pi_1, \pi_2, \ldots, \pi_d) : X \to X''$ such that $X' \overset{\gamma}{\rightarrowtail} X \overset{\pi}{\twoheadrightarrow} X''$ is a short exact sequence of matrix factorizations. In particular,

$$0 \longrightarrow F'_k \xrightarrow{\gamma_k} F_k \xrightarrow{\pi_k} F''_k \longrightarrow 0$$

is a short exact sequence of S-modules for each $k \in \mathbb{Z}_d$. Since F''_k is free, this sequence is split and therefore γ_k is a split injection.

Conversely, suppose the homomorphisms $\gamma_1, \ldots, \gamma_d$ are each split injections. For $k \in \mathbb{Z}_d$, set $F''_k \coloneqq \operatorname{cok} \gamma_k$ and $\pi_k : F_k \to F''_k$ the natural projection map. Notice that F''_k is a free

S-module of rank equal to $\operatorname{rank} F_k - \operatorname{rank} F'_k$. Now, for each $k \in \mathbb{Z}_d$, there exists a map $\varphi''_k : F_{k+1} \to F_k$ such that the following diagram with split exact rows commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'_1 & \xrightarrow{\gamma_1} & F_1 & \xrightarrow{\pi_1} & F''_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'_d} & & \downarrow{\scriptstyle \varphi_d} & & \vdots{\scriptstyle \varphi''_d} & & \\
0 & \longrightarrow & F'_d & \xrightarrow{\gamma_d} & F_d & \xrightarrow{\pi_d} & F''_d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'_{d-1}} & & \downarrow{\scriptstyle \varphi_{d-1}} & & \vdots{\scriptstyle \varphi''_{d-1}} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \varphi'_2} & & \downarrow{\scriptstyle \varphi_2} & & \vdots{\scriptstyle \varphi''_2} & & \\
0 & \longrightarrow & F'_2 & \xrightarrow{\gamma_2} & F_2 & \xrightarrow{\pi_2} & F''_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'_1} & & \downarrow{\scriptstyle \varphi_1} & & \vdots{\scriptstyle \varphi''_1} & & \\
0 & \longrightarrow & F'_1 & \xrightarrow{\gamma_1} & F_1 & \xrightarrow{\pi_1} & F''_1 & \longrightarrow & 0.
\end{array}$$

In particular, there exists $t_1 : F''_1 \to F_1$ such that $\pi_1 t_1 = 1_{F''_1}$. The splitting allows us to compute the composition along the right most column:

$$\begin{aligned}
\varphi''_1 \varphi''_2 \cdots \varphi''_d &= \pi_1 \varphi_1 \varphi_2 \cdots \varphi_d t_1 \\
&= f \cdot \pi_1 t_1 \\
&= f \cdot 1_{F''_1}.
\end{aligned}$$

Since the free modules $F''_1, F''_2, \ldots, F''_d$ are all of the same rank, we have that $X'' = (\varphi''_1 : F''_2 \to F''_1, \ldots, \varphi''_d : F''_1 \to F''_d) \in \mathrm{MF}^d_S(f)$ and

$$X' \xrightarrow{\ \gamma\ } X \xrightarrow{(\pi_1, \ldots, \pi_d)} X''$$

is a short exact sequence of matrix factorizations.

□

Lemma 2.2.3 indicates that not every monomorphism of matrix factorizations is an admissible monomorphism. The simplest example of this arises when $d = 2$.

Example 2.2.4. Suppose $(\varphi : G \to F, \psi : F \to G) \in \mathrm{MF}_S^2(f)$ with $\operatorname{cok}\psi \neq 0$. Then the tuple $(\psi, 1_G)$ forms a morphism between the matrix factorizations $(\varphi, \psi) \to (f \cdot 1_G, 1_G)$. This morphism is a monomorphism, in the sense that it can be cancelled on the left, but it is not admissible since the cokernel of ψ is not a free S-module.

The same is true of epimorphisms, that is, there are epimorphisms that are not admissible. For $(\varphi, \psi) \in \mathrm{MF}_S^2(f)$ with $\operatorname{cok}\psi \neq 0$, the tuple $(1_F, \psi)$ forms a morphism between the matrix factorizations $(f \cdot 1_F, 1_F) \to (\varphi, \psi)$. If $(a, b) \circ (1_F, \psi) = (a', b') \circ (1_F, \psi)$ for some morphisms $(a, b), (a', b')$, then $a = a'$ and $b\psi = b'\psi$. We can pre-compose both sides of the second equation with φ to get $b \cdot f = b' \cdot f$, hence $b = b'$. So, $(1_F, \psi)$ can be cancelled on the right but is not admissible epimorphism since ψ is not surjective.

Actually, further inspection of these examples shows they are both monomorphisms and both epimorphisms but neither is admissible of either type. In particular, neither is an isomorphism. In Abelian category, a monomorphism which is also an epimorphism must be an isomorphism. Similar examples can be constructed for all $d > 2$ and therefore we note that the category $\mathrm{MF}_S^d(f)$ is not Abelian for any $d \geq 2$.

Proposition 2.2.5. *The collection $\mathcal{E}_d$ of short exact sequences of matrix factorizations in $\mathrm{MF}_S^d(f)$ satisfies the axioms* (E2) *and* (E2$^{\mathrm{op}}$) .

Proof. We will show that (E2) holds. The proof that (E2$^{\mathrm{op}}$) is satisfied is similar. Suppose we have a diagram in $\mathrm{MF}_S^d(f)$

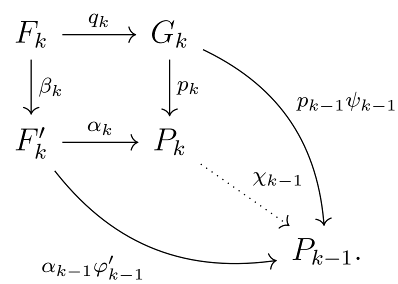
$$\begin{array}{ccc} X & \overset{q}{\rightarrowtail} & Y \\ \downarrow{\scriptstyle \beta} & & \\ X' & & \end{array} \tag{2.2.1}$$

where $Y = (\psi_1 : G_2 \to G_1, \ldots, \psi_d : G_1 \to G_d)$, $\beta = (\beta_1, \ldots, \beta_d)$, and $q = (q_1, \ldots, q_d)$. Let $k \in \mathbb{Z}_d$. We may take the push-out of the injection q_k and the map β_k which yields the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_k & \xrightarrow{q_k} & G_k & \longrightarrow & \operatorname{cok} q_k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_k} & & \downarrow{\scriptstyle p_k} & & \| & & \\ 0 & \longrightarrow & F'_k & \xrightarrow{\alpha_k} & P_k & \longrightarrow & \operatorname{cok} q_k & \longrightarrow & 0. \end{array}$$

We make the following observations from this diagram: Since the morphism q is an admissible monomorphism, the map q_k is a split injection. Hence, $\operatorname{cok} q_k$ is a free S-module. It follows that the bottom sequence also splits and so P_k is free with $\operatorname{rank}_S P_k = \operatorname{rank}_S F'_k + \operatorname{rank}_S G_k - \operatorname{rank}_S F_k$. This also implies that α_k is a split injection. Since k was arbitrary, this yields d free S-modules $P_1, P_2, \dots, P_d$, each of the same rank, and d-tuples $\alpha = (\alpha_1, \dots, \alpha_d)$ and $p = (p_1, \dots, p_d)$.

Next, let $k \in \mathbb{Z}_d$ and consider the diagram



There is a unique homomorphism $\chi_{k-1} : P_k \to P_{k-1}$ depicted above since P_k is the pushout of q_k and β_k and

$$p_{k-1}\psi_{k-1}q_k = p_{k-1}q_{k-1}\varphi_{k-1} = \alpha_{k-1}\beta_{k-1}\varphi_{k-1} = \alpha_{k-1}\varphi'_{k-1}\beta_k.$$

In particular, the map χ_{k-1} is given by

$$\chi_{k-1}(\overline{(a_{k'}, b_k)}) = \alpha_{k-1}\varphi'_{k-1}(a_k) + p_{k-1}\psi_{k-1}(b_k) = \overline{(\varphi'_{k-1}(a_k), \psi_{k-1}(b_k))} \in P_{k-1},$$

for any $(a_k, b_k) \in F'_k \oplus G_k$. In other words, χ_{k-1} is the map induced by the direct sum $\varphi'_{k-1} \oplus \psi_{k-1}$ on the quotients $P_k \to P_{k-1}$. These maps link together to form a sequence of compositions

$$P_1 \xrightarrow{\chi_d} P_d \xrightarrow{\chi_{d-1}} \cdots \xrightarrow{\chi_2} P_2 \xrightarrow{\chi_1} P_1.$$

From the explicit description of χ_k we have that $\chi_1\chi_2\cdots\chi_d = f \cdot 1_{P_1}$. Since the free S-modules $P_1, \dots, P_d$ are all of the same rank, it follows that $Y' = (\chi_1 : P_2 \to P_1, \cdots, \chi_d : P_1 \to P_d)$ is a matrix factorization of f with d factors.

It is not hard to see that $\alpha : X' \to Y'$ and $p : Y \to Y'$ form morphisms of matrix factorizations and that these morphisms render (2.2.1) a commutative square. As we showed above, the map α_k is a split injection for all $k \in \mathbb{Z}_d$. Hence, α is an admissible monomorphism by Lemma 2.2.3. To finish the proof, it suffices to check the necessary universal property which we omit as it is also a straightforward computation. □

Corollary 2.2.6. *The pair* $(\mathrm{MF}_S^d(f), \mathcal{E}_d)$ *is an exact category.* □

With the exact structure on $\mathrm{MF}_S^d(f)$ fixed, we will often omit reference to $\mathcal{E}_d$. We proceed now with the main goal of this section: to show that the exact category $\mathrm{MF}_S^d(f)$ is a Frobenius category. First, we recall the necessary definitions which can also be found in [Büh10].

2.2.1 Projective and injective objects

An object in an exact category $(\mathcal{A}, \mathcal{E})$ is called *projective*, respectively *injective*, if it satisfies the usual lifting property with respect to admissible epimorphisms, respectively admissible monomorphisms. The pair $(\mathcal{A}, \mathcal{E})$ is said to have *enough projectives* if for every object $X \in \mathcal{A}$, there is an admissible epimorphism $P \twoheadrightarrow X$ with P projective. Dually, $(\mathcal{A}, \mathcal{E})$ has *enough injectives* if for every object $X \in \mathcal{A}$, there is an admissible monomorphism $X \rightarrowtail I$ with I injective. The exact category $(\mathcal{A}, \mathcal{E})$ is said to be a *Frobenius category* if it has enough projectives, enough injectives, and the classes of projective objects and injective objects coincide.

Definition 2.2.7. For each $i \in \mathbb{Z}_d$, let $\mathcal{P}_i$ denote the matrix factorization of size 1 whose i-th component is multiplication by f on S while the rest are the identity on S. In other words, $\mathcal{P}_i$ is given by the composition

$$S_1 \xrightarrow{1} S_d \xrightarrow{1} \cdots \xrightarrow{1} S_{i+1} \xrightarrow{f} S_i \xrightarrow{1} \cdots \xrightarrow{1} S_2 \xrightarrow{1} S_1$$

where $S_k = S$ for each $k \in \mathbb{Z}_d$. We also set $\mathcal{P} = \bigoplus_{i \in \mathbb{Z}_d} \mathcal{P}_i$.

Lemma 2.2.8. *Let $X \in \mathrm{MF}_S^d(f)$ and $j \in \mathbb{Z}_d$. Then X is projective (respectively injective) if and only if $T^j(X)$ is projective (respectively injective).*

Proof. Suppose X is projective and $j \in \mathbb{Z}_d$. Let $\alpha = (\alpha_1, \ldots, \alpha_d) : X' \to X''$ be an admissible epimorphism and let $p = (p_1, \ldots, p_d) : T^j(X) \to X''$ be any morphism. Then we have morphisms $T^{-j}(\alpha) : T^{-j}(X') \to T^{-j}(X'')$ and $T^{-j}(p) : X \to T^{-j}(X'')$. The characterization of admissible epimorphisms in Lemma 2.2.3 implies that $T^{-j}(\alpha)$ is also an admissible epimorphism. Since X is projective, there exists $q = (q_1, q_2, \ldots, q_d) : X \to T^{-j}(X')$ such that $T^{-j}(\alpha)q = T^{-j}(p)$. Applying T^j we find that $\alpha T^j(q) = p$ implying that $T^j(X)$ is projective. The proof of the converse and both directions regarding injectivity are similar. ☐

Lemma 2.2.9. *The matrix factorizations $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_d$ are projective and injective objects in $\mathrm{MF}_S^d(f)$.*

Proof. Directly from the definition we see that $T^j(\mathcal{P}_i) = \mathcal{P}_{i-j}$ for any $i, j \in \mathbb{Z}_d$. Therefore, because of Lemma 2.2.8, it suffices to show that $\mathcal{P}_1$ is both projective and injective. We start by showing that $\mathcal{P}_1$ is projective.

Suppose $\alpha = (\alpha_1, \ldots, \alpha_d) : X \to X''$ is an admissible epimorphism and $p = (p_1, \ldots, p_d) : \mathcal{P}_1 \to X''$ is an arbitrary morphism. We need to complete the diagram

$$\begin{array}{ccc} & & \mathcal{P}_1 \\ & \overset{q}{\swarrow} & \downarrow p \\ X & \overset{\alpha}{\twoheadrightarrow} & X'' \end{array} \tag{2.2.2}$$

with a morphism q making the triangle commute. One component of this diagram is the following diagram of free S-modules

$$\begin{array}{ccccc} & & S & & \\ & \overset{q_1}{\swarrow} & \downarrow p_1 & & \\ F_1 & \xrightarrow{\alpha_1} & F_1'' & \longrightarrow & 0. \end{array}$$

Since S is free and α_1 is surjective, there exists a map $q_1 : S \to F_1$ such that $\alpha_1 q_1 = p_1$. We can use this map to construct a morphism of matrix factorizations $\mathcal{P}_1 \to X$ which makes (2.2.2) commute. Let $q = (q_1, \varphi_2\varphi_3 \cdots \varphi_d q_1, \varphi_3 \cdots \varphi_d q_1, \ldots, \varphi_d q_1)$. The fact that q forms a morphism $\mathcal{P}_1 \to X$ can be seen in the following diagram of S-modules

$$\begin{array}{ccccccccccc}
S & \xrightarrow{1} & S & \xrightarrow{1} & \cdots & \xrightarrow{1} & S & \xrightarrow{1} & S & \xrightarrow{f} & S \\
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle \varphi_d q_1} & & & & \downarrow{\scriptstyle \varphi_3 \cdots \varphi_d q_1} & & \downarrow{\scriptstyle \varphi_2\varphi_3 \cdots \varphi_d q_1} & & \downarrow{\scriptstyle q_1} \\
F_1 & \xrightarrow{\varphi_d} & F_d & \xrightarrow{\varphi_{d-1}} & \cdots & \xrightarrow{\varphi_3} & F_3 & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\varphi_1} & F_1.
\end{array}$$

Finally,

$$\begin{aligned}
\alpha q &= (\alpha_1 q_1, \alpha_2\varphi_2 \cdots \varphi_d q_1, \ldots, \alpha_d \varphi_d q_1) \\
&= (p_1, \varphi_2'' \cdots \varphi_d'' \alpha_1 q_1, \cdots, \varphi_d'' \alpha_1 q_1) \\
&= (p_1, \varphi_2'' \cdots \varphi_d'' p_1, \cdots, \varphi_d'' p_1) \\
&= (p_1, p_2, \ldots, p_d) \\
&= p
\end{aligned}$$

which implies that $\mathcal{P}_1$ is projective.

In order to show that $\mathcal{P}_1$ is an injective matrix factorization, let $\beta = (\beta_1, \ldots, \beta_d) : X' \rightarrowtail X$ be an admissible monomorphism and $a = (a_1, \ldots, a_d) : X' \to \mathcal{P}_1$ be an arbitrary morphism. We need to complete the diagram

$$\begin{array}{ccc}
X' & \overset{\beta}{\rightarrowtail} & X \\
\downarrow{\scriptstyle a} & \swarrow{\scriptstyle b} & \\
\mathcal{P}_1 & &
\end{array}.$$

Since β is an admissible monomorphism, each component β_k is split. In particular, there exists a map $t : F_2' \to F_2$ such that $t\beta_2 = 1_{F_2}$. This splitting allows us to build the morphism

$$b = (a_2 t \varphi_2' \cdots \varphi_d', a_2 t, a_2 t \varphi_2', \ldots, a_2 t \varphi_2' \cdots \varphi_{d-1}') : X \to \mathcal{P}_1$$

and this morphism satisfies $b\beta = a$. $\square$

In Lemma 2.3.2, we will see that $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_d$ are the only indecomposable projective (respectively injective) matrix factorizations up to isomorphism.

The next step is to show that $\mathrm{MF}_S^d(f)$ has enough projectives and enough injectives. Along the way, we construct the syzygy and cosyzygy of a matrix factorization and give explicit formulas for each.

2.2.2 The syzygy of a matrix factorization

In this section, we construct short exact sequences

$$K \rightarrowtail P \twoheadrightarrow X \quad \text{and} \quad X \rightarrowtail I \twoheadrightarrow K'$$

with P projective and I injective for any $X \in \mathrm{MF}_S^d(f)$. Then we give explicit formula for the resulting syzygy K and cosyzygy K' of X.

Construction 2.2.10. Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$ be of size n. Set $\widehat{F}_k = \bigoplus_{i=1}^{d-1} F_{k+i}$. For each $k \in \mathbb{Z}_d$, define S-homomorphisms $D_k : F_{k+1} \oplus \widehat{F}_{k+1} \to F_k \oplus \widehat{F}_k$ and $D'_k : F_{k+1} \oplus \widehat{F}_{k+1} \to F_k \oplus \widehat{F}_k$ by

$$D_k(a_{k+1}, a_{k+2}, \ldots, a_{k-1}, a_k) = (fa_k, a_{k+1}, \ldots, a_{k-1})$$

and

$$D'_k(a_{k+1}, a_{k+2}, \ldots, a_{k-1}, a_k) = (a_k, fa_{k+1}, a_{k+2}, \ldots, a_{k-1})$$

for all $a_i \in F_i, i \in \mathbb{Z}_d$. In other words, D_k and D'_k are the $d \times d$ block matrices

$$D_k = \begin{pmatrix} 0 & 0 & \dots & 0 & f \cdot 1_{F_k} \\ 1_{F_{k+1}} & 0 & \dots & 0 & 0 \\ 0 & 1_{F_{k+2}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{F_{k-1}} & 0 \end{pmatrix} \quad \text{and} \quad D'_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 1_{F_k} \\ f \cdot 1_{F_{k+1}} & 0 & \dots & 0 & 0 \\ 0 & 1_{F_{k+2}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{F_{k-1}} & 0 \end{pmatrix}.$$

Set $P(X) = (D_1, D_2, \dots, D_d)$ and $I(X) = (D'_1, D'_2, \dots, D'_d)$. Then the d-tuples $P(X)$ and $I(X)$ form matrix factorizations of f both isomorphic to $\bigoplus_{i=1}^d \mathcal{P}_i^n$.

For each $i, k \in \mathbb{Z}_d$, define a homomorphism $\theta_{ki}^X : F_i \to F_k$ given by

$$\theta_{ki}^X = \begin{cases} 1_{F_k} & i = k \\ \varphi_k \varphi_{k+1} \cdots \varphi_{i-2} \varphi_{i-1} & i \neq k. \end{cases}$$

Then, for each $k \in \mathbb{Z}_d$, define $\Theta_k^X : \widehat{F}_k \to F_k$ and $\Xi_k^X : F_k \to \widehat{F}_k$ by

$$\Theta_k^X(a_{k+1}, a_{k+2}, \dots, a_{k-1}) = \sum_{i \neq k} \theta_{ki}^X(a_i)$$

and

$$\Xi_k^X(a_k) = \left(\theta_{(k+1)k}^X(a_k), \theta_{(k+2)k}^X(a_k), \dots, \theta_{(k-1)k}^X(a_k)\right).$$

Let $k \in \mathbb{Z}_d$ and consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{F}_{k+1} & \xrightarrow{\epsilon_{k+1}^X} & F_{k+1} \oplus \widehat{F}_{k+1} & \xrightarrow{\rho_{k+1}^X} & F_{k+1} & \longrightarrow & 0 \\ & & \Big\downarrow \Omega_k & & \Big\downarrow D_k & & \Big\downarrow \varphi_k & & \\ 0 & \longrightarrow & \widehat{F}_k & \xrightarrow{\epsilon_k^X} & F_k \oplus \widehat{F}_k & \xrightarrow{\rho_k^X} & F_k & \longrightarrow & 0 \end{array} \tag{2.2.3}$$

where $\rho_k^X = \begin{pmatrix} 1_{F_k} & \Theta_k^X \end{pmatrix}$ and $\epsilon_k^X = \begin{pmatrix} -\Theta_k^X \\ \\ 1_{\widehat{F}_k} \end{pmatrix}$. The rows are split exact sequences of free S-modules and one can check that right most square commutes by recalling that $\varphi_k\theta_{(k+1)i} = \theta_{ki}$ for $i \neq k$ and $\varphi_k\theta_{(k+1)k} = f \cdot 1_{F_k}$. Thus, there is an induced map $\Omega_k : \widehat{F}_{k+1} \to \widehat{F}_k$ as depicted. Since the rows are split, Ω_k can be computed by using the splitting, that is, $\Omega_k = \pi_k D_k \epsilon_{k+1}$ where $\pi_k : F_k \oplus \widehat{F}_k \to \widehat{F}_k$ is projection onto $\widehat{F}_k$. In particular,

$$\begin{aligned}
\Omega_k(a_{k+2}, a_{k+3}, \ldots, a_{k-1}, a_k) &= \pi_k D_k \epsilon_{k+1}(a_{k+2}, a_{k+3}, \ldots, a_{k-1}, a_k) \\
&= \pi_k D_k \left(-\sum_{i \neq k+1} \theta_{(k+1)i}(a_i), a_{k+2}, \ldots, a_{k-1}, a_k \right) \\
&= \pi_k \left(f a_k, -\sum_{i \neq k+1} \theta_{(k+1)i}(a_i), a_{k+2}, \ldots, a_{k-1} \right) \\
&= \left(-\sum_{i \neq k+1} \theta_{(k+1)i}(a_i), a_{k+2}, \ldots, a_{k-1} \right)
\end{aligned}$$

and therefore we can represent the components of Ω_k as

$$\Omega_k = \begin{pmatrix}
-\theta_{(k+1)(k+2)} & -\theta_{(k+1)(k+3)} & -\theta_{(k+1)(k+4)} & \cdots & -\theta_{(k+1)(k-1)} & -\theta_{(k+1)(k)} \\
1_{F_{k+2}} & 0 & 0 & \cdots & 0 & 0 \\
0 & 1_{F_{k+3}} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & 1_{F_{k+4}} & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1_{F_{k-1}} & 0
\end{pmatrix}.$$

Since k was arbitrary, we have a d-tuple $(\Omega_1, \Omega_2, \ldots, \Omega_d)$ which has the property that

$$\Omega_1\Omega_2 \cdots \Omega_d = \pi_1 D_1 D_2 \cdots D_d \epsilon_1 = f\pi_1\epsilon_1 = f \cdot 1_{\widehat{F}_1}.$$

Since the free S-modules $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_d$ are all of the same rank, it follows that $(\Omega_1, \ldots, \Omega_d) \in \mathrm{MF}_S^d(f)$. We denote this matrix factorization by $\Omega_{\mathrm{MF}_S^d(f)}(X)$ and refer to it as the *syzygy* of

X. Combining the diagrams (2.2.3) for all $k \in \mathbb{Z}_d$ we have a short exact sequence

$$\Omega_{\mathrm{MF}_S^d(f)}(X) \overset{\epsilon}{\rightarrowtail} P(X) \overset{\rho}{\twoheadrightarrow} X \tag{2.2.4}$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_d)$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_d)$.

Similarly, we have a matrix factorization $\Omega^-_{\mathrm{MF}_S^d(f)}(X) = (\Omega_1^-, \Omega_2^-, \ldots, \Omega_d^-)$, the *cosyzygy* of X, and a short exact sequence of matrix factorizations induced by the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{k+1} & \xrightarrow{\lambda^X_{k+1}} & F_{k+1} \oplus \widehat{F}_{k+1} & \xrightarrow{\eta^X_{k+1}} & \widehat{F}_{k+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle D'_k} & & \vdots{\scriptstyle \Omega^-_k} & & \\ 0 & \longrightarrow & F_k & \xrightarrow{\lambda^X_k} & F_k \oplus \widehat{F}_k & \xrightarrow{\eta^X_k} & \widehat{F}_k & \longrightarrow & 0, \end{array} \tag{2.2.5}$$

for all $k \in \mathbb{Z}_d$, where $\eta^X_k = \begin{pmatrix} -\Xi^X_k & 1_{\widehat{F}_k} \end{pmatrix}$, $\lambda^X_k = \begin{pmatrix} 1_{F_k} \\ \Xi^X_k \end{pmatrix}$. The induced short exact sequence is

$$X \overset{\lambda}{\rightarrowtail} I(X) \overset{\eta}{\twoheadrightarrow} \Omega^-_{\mathrm{MF}_S^d(f)}(X), \tag{2.2.6}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_d)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$. Finally, we can represent the components of Ω^-_k by

$$\Omega^-_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -\theta_{(k+1)k} \\ 1_{F_{k+2}} & 0 & \cdots & 0 & 0 & -\theta_{(k+2)k} \\ 0 & 1_{F_{k+3}} & \ddots & 0 & 0 & -\theta_{(k+3)k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{F_{k-2}} & 0 & -\theta_{(k-2)k} \\ 0 & 0 & \cdots & 0 & 1_{F_{k-1}} & -\theta_{(k-1)k} \end{pmatrix}.$$

Example 2.2.11. Let $(\varphi, \psi) \in \mathrm{MF}_S^2(f)$. Then

$$\Omega_{\mathrm{MF}_S^2(f)}(\varphi, \psi) = (-\psi, -\varphi) \cong (\psi, \varphi) = T(\varphi, \psi)$$

and

$$\Omega^-_{\mathrm{MF}^2_S(f)}(\varphi,\psi) = (-\psi,-\varphi) \cong (\psi,\varphi) = T(\varphi,\psi).$$

In particular, both $\Omega_{\mathrm{MF}^2_S(f)}(-)$ and $\Omega^-_{\mathrm{MF}^2_S(f)}(-)$ are isomorphic to the shift functor when $d=2$. In this case, (φ,ψ) is reduced if and only if $\Omega_{\mathrm{MF}^2_S(f)}(\varphi,\psi)$ (respectively $\Omega^-_{\mathrm{MF}^2_S(f)}(\varphi,\psi)$) is reduced. However, for $X \in \mathrm{MF}^d_S(f)$ with $d>2$, neither $\Omega_{\mathrm{MF}^2_S(f)}(X)$ nor $\Omega^-_{\mathrm{MF}^2_S(f)}(X)$ will be reduced. For instance, if $X = (\varphi_1 : F_2 \to F_1, \varphi_2 : F_3 \to F_2, \varphi_3 : F_1 \to F_3) \in \mathrm{MF}^3_S(f)$ is of size n, then

$$\Omega_{\mathrm{MF}^3_S(f)}(X) = \left(\begin{pmatrix} -\varphi_2 & -\varphi_2\varphi_3 \\ 1_{F_3} & 0 \end{pmatrix}, \begin{pmatrix} -\varphi_3 & -\varphi_3\varphi_1 \\ 1_{F_1} & 0 \end{pmatrix}, \begin{pmatrix} -\varphi_1 & -\varphi_1\varphi_2 \\ 1_{F_2} & 0 \end{pmatrix} \right)$$

and

$$\Omega^-_{\mathrm{MF}^3_S(f)}(X) = \left(\begin{pmatrix} 0 & -\varphi_2\varphi_3 \\ 1_{F_3} & -\varphi_3 \end{pmatrix}, \begin{pmatrix} 0 & -\varphi_3\varphi_1 \\ 1_{F_1} & -\varphi_1 \end{pmatrix}, \begin{pmatrix} 0 & -\varphi_1\varphi_2 \\ 1_{F_2} & -\varphi_2 \end{pmatrix} \right)$$

which are both of size $2n$.

One can also write down the short exact sequence of matrix factorizations defining $\Omega_{\mathrm{MF}^d_S(f)}(X)$. For example, in the case $d=3$ we have

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
F_2 \oplus F_3 & \xrightarrow{\begin{pmatrix} -\varphi_1 & -\varphi_1\varphi_2 \\ 1_{F_2} & 0 \end{pmatrix}} & F_1 \oplus F_2 & \xrightarrow{\begin{pmatrix} -\varphi_3 & -\varphi_3\varphi_1 \\ 1_{F_1} & 0 \end{pmatrix}} & F_3 \oplus F_1 & \xrightarrow{\begin{pmatrix} -\varphi_2 & -\varphi_2\varphi_3 \\ 1_{F_3} & 0 \end{pmatrix}} & F_2 \oplus F_3 \\
\Big\downarrow {\scriptsize\begin{pmatrix} -\varphi_1 & -\varphi_1\varphi_2 \\ 1_{F_2} & 0 \\ 0 & 1_{F_3} \end{pmatrix}} & & \Big\downarrow {\scriptsize\begin{pmatrix} -\varphi_3 & -\varphi_3\varphi_1 \\ 1_{F_1} & 0 \\ 0 & 1_{F_2} \end{pmatrix}} & & \Big\downarrow {\scriptsize\begin{pmatrix} -\varphi_2 & -\varphi_2\varphi_3 \\ 1_{F_3} & 0 \\ 0 & 1_{F_1} \end{pmatrix}} & & \Big\downarrow {\scriptsize\begin{pmatrix} -\varphi_1 & -\varphi_1\varphi_2 \\ 1_{F_2} & 0 \\ 0 & 1_{F_3} \end{pmatrix}} \\
F_1 \oplus F_2 \oplus F_3 & \xrightarrow[D_3]{} & F_3 \oplus F_1 \oplus F_2 & \xrightarrow[D_2]{} & F_2 \oplus F_3 \oplus F_1 & \xrightarrow[D_1]{} & F_1 \oplus F_2 \oplus F_3 \\
\Big\downarrow {\scriptsize\begin{pmatrix} 1_{F_1} & \varphi_1 & \varphi_1\varphi_2 \end{pmatrix}} & & \Big\downarrow {\scriptsize\begin{pmatrix} 1_{F_3} & \varphi_3 & \varphi_3\varphi_1 \end{pmatrix}} & & \Big\downarrow {\scriptsize\begin{pmatrix} 1_{F_2} & \varphi_2 & \varphi_2\varphi_3 \end{pmatrix}} & & \Big\downarrow {\scriptsize\begin{pmatrix} 1_{F_1} & \varphi_1 & \varphi_1\varphi_2 \end{pmatrix}} \\
F_1 & \xrightarrow{\varphi_3} & F_3 & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\varphi_1} & F_1 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
0 & & 0 & & 0 & & 0.
\end{array}$$

Another observation that can be made from these formulas is that, for each $k \in \mathbb{Z}_3$, we have an isomorphism of R-modules

$$\operatorname{cok}\begin{pmatrix} -\varphi_{k+1} & -\varphi_{k+1}\varphi_{k+2} \\ 1_{F_{k+2}} & 0 \end{pmatrix} \cong \operatorname{cok}(\varphi_{k+1}\varphi_{k+2}) \cong \operatorname{syz}^1_R(\operatorname{cok}\varphi_k) \oplus R^{m_k}$$

for some $m_k \geq 0$. A similar statement is true for $\Omega^-_{\mathrm{MF}^3_S(f)}(X)$ and more generally we have the following proposition.

Proposition 2.2.12. *Let $X \in \mathrm{MF}^d_S(f)$ be of size n. Let $\Omega_{\mathrm{MF}^d_S(f)}(X)$ and $\Omega^-_{\mathrm{MF}^d_S(f)}(X)$ be the matrix factorizations constructed in 2.2.10. Then, for each $k \in \mathbb{Z}_d$,*

$$\operatorname{cok}(\Omega_k) \cong \operatorname{syz}^1_R(\operatorname{cok}\varphi_k) \oplus R^{m_k} \cong \operatorname{cok}(\Omega^-_k)$$

where $m_k = n - \mu_R(\operatorname{cok}\varphi_k)$.

Proof. Let $k \in \mathbb{Z}_d$. The diagram (2.2.3) induces the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{F}_{k+1} & \xrightarrow{\Omega_k} & \widehat{F}_k & \longrightarrow & \operatorname{cok}\Omega_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{k+1} \oplus \widehat{F}_{k+1} & \xrightarrow{D_k} & F_k \oplus \widehat{F}_k & \longrightarrow & F_k/fF_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{k+1} & \xrightarrow{\varphi_k} & F_k & \longrightarrow & \operatorname{cok}\varphi_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. The right most column displays $\operatorname{cok}\Omega_k$ as a (not necessarily reduced) syzygy of $\operatorname{cok}\varphi_k$ over R as desired.

Alternatively, we can see from the explicit formulas for Ω_k and Ω_k^- that

$$\operatorname{cok}\Omega_k \cong \operatorname{cok}\theta^X_{(k+1)k} \cong \operatorname{cok}\Omega_k^-.$$

Now, recall that $\theta_{(k+1)k} = \varphi_{k+1}\varphi_{k+2}\cdots\varphi_{k-1}$. Since $(\varphi_k, \theta_{(k+1)k})$ is a matrix factorization with 2 factors, we have that $\operatorname{cok}(\theta_{(k+1)k}) \cong \operatorname{syz}^1_R(\operatorname{cok}\varphi_k) \oplus R^{m_k}$ for some $m_k \geq 0$. In particular, $m_k = n - \mu_R(\operatorname{cok}\varphi_k)$ by the uniqueness of minimal free resolutions over R. □

2.2.3 Frobenius structure

We note that since the matrix factorizations $P(X) \cong I(X) \cong \bigoplus_{i=1}^d \mathcal{P}_i^n$ are projective and injective by Lemma 2.2.9, the sequences (2.2.4) and (2.2.6) imply that $\mathrm{MF}^d_S(f)$ has enough projectives and enough injectives. Additionally, we have the following.

Proposition 2.2.13. *An object $X \in \mathrm{MF}^d_S(f)$ is projective if and only if it is injective.*

Proof. Let $X \in \mathrm{MF}^d_S(f)$. If X is projective, (2.2.4) implies that X is a summand of the injective matrix factorization $P(X)$ and therefore is injective. Conversely, if X is injective, (2.2.6) implies it is a summand of the projective $I(X)$. □

We have therefore established our original goal of this section.

Theorem 2.2.14. *The category $\mathrm{MF}^d_S(f)$ is a Frobenius category.* □

For matrix factorizations $X, X' \in \mathrm{MF}^d_S(f)$, let $I(X, X')$ denote the set of morphisms X to X' that factor through an injective (equivalently a projective) matrix factorization. The stable category $\underline{\mathrm{MF}}^d_S(f)$ is formed by taking the same objects as $\mathrm{MF}^d_S(f)$ and morphisms given by the quotient

$$\operatorname{Hom}_{\underline{\mathrm{MF}}^d_S(f)}(X, X') = \operatorname{Hom}_{\mathrm{MF}^d_S(f)}(X, X')/I(X, X').$$

A consequence of Theorem 2.2.14 is that $\underline{\mathrm{MF}}^d_S(f)$ carries the structure of a triangulated category with suspension functor given by $\Omega^-_{\mathrm{MF}^d_S(f)}(-)$ [Hap88, p. I.2]. We call a morphism

$(\alpha_1, \alpha_2, \dots, \alpha_d) : X \to X'$ *null-homotopic* if there exist S-homomorphisms $s_j : F_j \to F'_{j-1}$, $j \in \mathbb{Z}_d$, such that

$$\alpha_i = \sum_{k \in \mathbb{Z}_d} \theta^{X'}_{i(i-k)} s_{i-k+1} \theta^{X}_{(i-k+1)i}$$

for each $i \in \mathbb{Z}_d$. We denote by $\mathrm{HMF}^d_S(f)$ the *homotopy category of matrix factorizations* which has the same objects as $\mathrm{MF}^d_S(f)$ and, for any $X, X' \in \mathrm{MF}^d_S(f)$, has morphisms

$$\mathrm{Hom}_{\mathrm{HMF}^d_S(f)}(X, X') = \mathrm{Hom}_{\mathrm{MF}^d_S(f)}(X, X')/\sim,$$

where $\sim$ is the equivalence relation $\alpha \sim \alpha'$ if and only if $\alpha - \alpha'$ is null-homotopic.

Proposition 2.2.15. *The stable category* $\underline{\mathrm{MF}}^d_S(f)$ *and the homotopy category* $\mathrm{HMF}^d_S(f)$ *coincide.*

Proof. Let $X, X' \in \mathrm{MF}^d_S(f)$. It suffices to show that a morphism $\alpha : X \to X'$ is null-homotopic if and only if it factors through the morphism $\lambda^X : X \to I(X)$. The proof relies on an explicit description of morphisms $I(X) \to X'$. Indeed, if $\beta : I(X) \to X'$ is any morphism, then, for any $k, j \in \mathbb{Z}_d$, we have a commutative diagram

$$\begin{array}{ccccccccc}
F_{k+j-1} \oplus \widehat{F}_{k+j-1} & \xrightarrow{D'_{k+j-2}} & F_{k+j-2} \oplus \widehat{F}_{k+j-2} & \xrightarrow{D'_{k+j-3}} & \cdots & \xrightarrow{D'_{k+1}} & F_{k+1} \oplus \widehat{F}_{k+1} & \xrightarrow{D'_{k}} & F_k \oplus \widehat{F}_k \\
\downarrow{\scriptstyle \beta_{k+j-1}} & & \downarrow{\scriptstyle \beta_{k+j-2}} & & & & \downarrow{\scriptstyle \beta_{k+1}} & & \downarrow{\scriptstyle \beta_{k}} \\
F'_{k+j-1} & \xrightarrow{\varphi'_{k+j-2}} & F'_{k+j-2} & \xrightarrow{\varphi'_{k+j-3}} & \cdots & \xrightarrow{\varphi'_{k+1}} & F'_{k+1} & \xrightarrow{\varphi'_{k}} & F'_k.
\end{array}$$

Recall that, for any $\ell \in \mathbb{Z}_d$ and $(a_{\ell+1}, a_{\ell+2}, \dots, a_{\ell-1}, a_\ell) \in F_{\ell+1} \oplus \widehat{F}_{\ell+1}$, we have that

$$D'_\ell((a_{\ell+1}, a_{\ell+2}, \dots, a_{\ell-1}, a_\ell)) = (a_\ell, f a_{\ell+1}, a_{\ell+2}, \dots, a_{\ell-1}).$$

In other words, D'_ℓ cyclically permutes each components of $F_{\ell+1} \oplus \widehat{F}_{\ell+1}$ but only scales $F_{\ell+1}$ by f.

Now, let $k, j \in \mathbb{Z}_d$ with $j \neq 1$. It follows that the maps $D'_k, D'_{k+1}, \dots, D'_{k+j-2}$ are the identity on F_{k+j} and therefore so is the product $D'_k D'_{k+1} \cdots D'_{k+j-2}$. Write the components of β_i as

$$\beta_i = \begin{pmatrix} \beta_{ii} & \beta_{i(i+1)} & \beta_{i(i+2)} & \cdots & \beta_{i(i-1)} \end{pmatrix}$$

for some $\beta_{i(i+s)} : F_{i+s} \to F'_i$. By the above observation, we have that

$$\beta_{k(k+j)} = \varphi'_k \varphi'_{k+1} \cdots \varphi'_{k+j-2} \beta_{(k+j-1)(k+j)} = \theta^{X'}_{k(k+j-1)} \beta_{(k+j-1)(k+j)}$$

by the commutativity of the outer most rectangle of the diagram above. Therefore,

$$\beta_k = \begin{pmatrix} \theta^{X'}_{k(k-1)} \beta_{(k-1)k} & \beta_{k(k+1)} & \theta^{X'}_{k(k+1)} \beta_{(k+1)(k+2)} & \cdots & \theta^{X'}_{k(k-2)} \beta_{(k-2)(k-1)} \end{pmatrix}.$$

Now, if $\beta : I(X) \to X'$ is such that $\beta \lambda^X = \alpha$, then

$$\alpha_k = \beta_k \lambda^X_k = -\sum_{i \in \mathbb{Z}_d} \theta^{X'}_{k(k-i)} \beta_{(k-i)(k-i+1)} \theta^X_{(k-i+1)k}$$

for any $k \in \mathbb{Z}_d$. This says precisely that α is null-homotopic via the maps $\{-\beta_{(j-1)j}\}_{j \in \mathbb{Z}_d}$.

Conversely, if α is null-homotopic via maps $s_j : F_j \to F'_{j-1}$, then it is not hard to see that the maps

$$\gamma_k = \begin{pmatrix} \theta^{X'}_{k(k-1)} s_k & s_{k+1} & \theta^{X'}_{k(k+1)} s_{k+2} & \cdots & \theta^{X'}_{k(k-1)} s_{k-1} \end{pmatrix},$$

for $k \in \mathbb{Z}_d$, form a morphism $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d) : I(X) \to X'$ such that $\alpha = \gamma \lambda^X$. □

To end this section, we give explicit formula for the mapping cone of a morphism. Suppose $\alpha : X \to X'$ is a morphism in $\mathrm{MF}^d_S(f)$. The *mapping cone* of α is the matrix factorization $C(\alpha) = (\Delta_1, \Delta_2, \ldots, \Delta_d)$ where

$$\Delta_k = \begin{pmatrix} \varphi'_k & 0 & 0 & \cdots & 0 & \alpha_k \\ 0 & 0 & 0 & \cdots & 0 & -\theta^X_{(k+1)k} \\ 0 & 1_{F_{k+2}} & 0 & \cdots & 0 & -\theta^X_{(k+2)k} \\ 0 & 0 & 1_{F_{k+3}} & \ddots & 0 & -\theta^X_{(k+3)k} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1_{F_{k-1}} & -\theta^X_{(k-1)k} \end{pmatrix} : F'_{k+1} \oplus \widehat{F}_{k+1} \to F'_k \oplus \widehat{F}_k$$

for all $k \in \mathbb{Z}_d$. The cone of α fits into a commutative diagram

$$\begin{array}{ccccc} X & \overset{\lambda^X}{\rightarrowtail} & I(X) & \overset{\eta^X}{\twoheadrightarrow} & \Omega^-_{\mathrm{MF}^d_S(f)}(X) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| \\ X' & \overset{q}{\rightarrowtail} & C(\alpha) & \overset{p}{\twoheadrightarrow} & \Omega^-_{\mathrm{MF}^d_S(f)}(X) \end{array}$$

where $p_k = \begin{pmatrix} 0 & 1_{\widehat{F}_k} \end{pmatrix}$, $q_k = \begin{pmatrix} 1_{F'_k} \\ 0 \end{pmatrix}$, and $\beta_k = \begin{pmatrix} \alpha_k & 0 \\ -\Xi^X_k & 1_{\widehat{F}_k} \end{pmatrix}$ for all $k \in \mathbb{Z}_d$, and $p = (p_1, \ldots, p_d)$, $q = (q_1, \ldots, q_d)$, and $\beta = (\beta_1, \ldots, \beta_d)$.

Remark 2.2.16. Let $X \in \mathrm{MF}^d_S(f)$. By tensoring with $R = S/(f)$, which we denote here by $\overline{\square} = \square \otimes_S R$, one can associate to X an infinite chain of free R-modules:

$$\cdots \xrightarrow{\overline{\varphi}_2} \overline{F}_2 \xrightarrow{\overline{\varphi}_1} \overline{F}_1 \xrightarrow{\overline{\varphi}_d} \overline{F}_d \xrightarrow{\overline{\varphi}_{d-1}} \cdots \xrightarrow{\overline{\varphi}_2} \overline{F}_2 \xrightarrow{\overline{\varphi}_1} \overline{F}_1 \xrightarrow{\overline{\varphi}_d} \cdots$$

In the case $d = 2$, this chain is an acyclic complex and, by truncating appropriately, it forms a free resolution of $\operatorname{cok} \varphi_1$ over R (or of $\operatorname{cok} \varphi_2$). However, if $d > 2$, this chain is not acyclic. In fact, it is not a complex. Instead, it is precisely an *acyclic d-complex* (see [IKM17]). With this perspective in mind, it is likely that the formulas given in this section can be obtained as lifted versions of the ones found in [IKM17, Section 2].

2.3 More on the structure of $\mathbf{MF}_S^d(f)$

For a matrix factorization $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d)$, we study its syzygy, $\Omega_{\mathrm{MF}_S^d(f)}(X) = (\Omega_1, \Omega_2, \ldots, \Omega_d)$, defined in Section 2.2.2. We start with a fundamental result regarding the relationship between projective summands in $\mathrm{MF}_S^d(f)$ and free summands in $\mathrm{MCM}(R)$. Recall that a matrix factorization $X \in \mathrm{MF}_S^d(f)$ is *stable* if, for all $k \in \mathbb{Z}_d$, the MCM R-module $\operatorname{cok} \varphi_k$ has no direct summands isomorphic to R.

Proposition 2.3.1. *Let $X = (\varphi_1, \ldots, \varphi_d) \in \mathrm{MF}_S^d(f)$ and set $M_i = \operatorname{cok} \varphi_i$ for all $i \in \mathbb{Z}_d$. Then X has a projective summand isomorphic to $\mathcal{P}_i$ if and only if M_i has a free R-summand.*

Proof. The statement holds when $d = 2$ (for instance see [Yos90, p. 7.5]). So, assume $d \geq 3$. One direction is immediate: If $X \cong X' \oplus \mathcal{P}_i$ for some $X' = (\varphi'_1, \ldots, \varphi'_d) \in \mathrm{MF}_S^d(f)$ and $i \in \mathbb{Z}_d$, then $M_i \cong \operatorname{cok} \varphi'_i \oplus R$.

A matrix factorization Y is a summand of X if and only if $T^j(Y)$ is a summand of $T^j(X)$ for any $j \in \mathbb{Z}_d$. Therefore, for the converse, we may assume $i = 1$. That is, assume $M_1 \cong M \oplus R$ for some MCM R-module M. By Proposition 1.2.8 (i), there exists $(\varphi : G \to F, \psi : F \to G) \in \mathrm{MF}_S^2(f)$, with φ minimal, such that $\operatorname{cok} \varphi \cong M$. Then

$$0 \longrightarrow G \oplus S \xrightarrow{\begin{pmatrix} \varphi & 0 \\ 0 & f \end{pmatrix}} F \oplus S \longrightarrow M_1 \longrightarrow 0$$

is a minimal free resolution of M_1 over S. Thus, there exists isomorphisms α and β and a commutative diagram

$$\begin{CD}
0 @>>> F_2 @>\varphi_1>> F_1 @>>> M_1 @>>> 0\\
@. @V\beta VV @V\alpha VV @| @.\\
0 @>>> G \oplus S \oplus S^m @>>{\begin{pmatrix} \varphi & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1_{S_m} \end{pmatrix}}> F \oplus S \oplus S^m @>>> M_1 @>>> 0
\end{CD}$$

for some $m \geq 0$. It follows that we have an isomorphism of matrix factorizations in $\mathrm{MF}_S^2(f)$:

$$(\varphi_1, \varphi_2\varphi_3 \cdots \varphi_d) \cong \left(\begin{pmatrix} \varphi & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1_{S_m} \end{pmatrix}, \begin{pmatrix} \psi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f \cdot 1_{S_m} \end{pmatrix} \right).$$

The isomorphisms α and β also give us an isomorphism of matrix factorizations in $\mathrm{MF}_S^d(f)$:

$$\begin{aligned} X &\cong (\alpha\varphi_1, \varphi_2, \ldots, \varphi_{d-1}, \varphi_d\alpha^{-1}) \\ &\cong (\alpha\varphi_1\beta^{-1}, \beta\varphi_2, \ldots \varphi_{d-1}, \varphi_d\alpha^{-1}). \end{aligned}$$

Let $p_1 : F \oplus S \oplus S^m \to S$ and $p_2 : G \oplus S \oplus S^m \to S$ be projection onto the middle components of $F \oplus S \oplus S^m$ and $G \oplus S \oplus S^m$ respectively. Consider the diagram

$$\begin{CD}
F \oplus S \oplus S^m @>\varphi_d\alpha^{-1}>> F_d @>\varphi_{d-1}>> \cdots @>\varphi_4>> F_4 @>\varphi_3>> F_3 @>\beta\varphi_2>> G \oplus S \oplus S^m @>\alpha\varphi_1\beta^{-1}>> F \oplus S \oplus S^m\\
@Vp_1VV @Vp_2\beta\varphi_2\varphi_3\cdots\varphi_{d-1}VV @. @Vp_2\beta\varphi_2\varphi_3VV @Vp_2\beta\varphi_2VV @Vp_2VV @Vp_1VV\\
S @>1>> S @>1>> \cdots @>1>> S @>1>> S @>1>> S @>f>> S.
\end{CD}$$

The two right most squares commute, the first since $\alpha\varphi_1\beta^{-1} = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1_{S_m} \end{pmatrix}$ and the second by construction. Similarly, for $k = 3, 4, \ldots, d-1$, the square

$$\begin{CD}
F_{k+1} @>\varphi_k>> F_k\\
@Vp_2\beta\varphi_2\varphi_3\cdots\varphi_{k-1}\varphi_kVV @VVp_2\beta\varphi_2\varphi_3\cdots\varphi_{k-1}V\\
S @>1>> S
\end{CD}$$

commutes. Since $p_1\alpha\varphi_1\beta^{-1} = fp_2$, we have that

$$fp_1 = p_1 f = p_1\alpha\varphi_1\beta^{-1}\beta\varphi_2\varphi_3\cdots\varphi_{d-1}\varphi_d\alpha^{-1}$$
$$= fp_2\beta\varphi_2\varphi_3\cdots\varphi_{d-1}\varphi_d\alpha^{-1}.$$

We may cancel f on the left to conclude that the left most square also commutes. Thus, we have a morphism

$$X \cong (\alpha\varphi_1\beta^{-1}, \beta\varphi_2, \ldots, \varphi_{d-1}, \varphi_d\alpha^{-1}) \to \mathcal{P}_1.$$

We claim that this morphism is an admissible epimorphism. By Lemma 2.2.3, it suffices to show that each of the vertical maps depicted above are surjective. By the commutativity of the diagram,

$$p_1 = (p_2\beta\varphi_2\varphi_3\cdots\varphi_k)(\varphi_{k+1}\varphi_{k+2}\cdots\varphi_{d-1}\varphi_d\alpha^{-1})$$

for each $k = 2, 3, \ldots, d-1$. Since p_1 is surjective, this implies $p_2\beta\varphi_2\varphi_3\cdots\varphi_k$ is surjective for each $k = 2, 3, \ldots, d-1$ as claimed. Since $\mathcal{P}_1$ is projective, the admissible epimorphism $X \twoheadrightarrow \mathcal{P}_1$ implies that X has a direct summand isomorphic to $\mathcal{P}_1$. □

Corollary 2.3.2.

(i) The objects $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_d \in \mathrm{MF}_S^d(f)$ are the only indecomposable projectives (equivalently injectives) up to isomorphism.

(ii) Let Q be a non-zero projective (equivalently injective) object in $\mathrm{MF}_S^d(f)$. Then $Q \cong \bigoplus_{i\in\mathbb{Z}_d} \mathcal{P}_i^{s_i}$ for some $s_i \geq 0$.

(iii) A matrix factorization $X \in \mathrm{MF}_S^d(f)$ is stable if and only if it has no non-zero projective direct summands.

Proof. Let $Q = (Q_1, Q_2, \ldots, Q_d) \in \mathrm{MF}_S^d(f)$ be an indecomposable projective of size n. Then we have a short exact sequence of the form (2.2.4), and more specifically, Q is a direct summand of $P(Q) \cong \bigoplus_{i\in\mathbb{Z}_d} \mathcal{P}_i^n$. It follows that, for any $k \in \mathbb{Z}_d$, $\operatorname{cok} Q_k$ is either 0 or has a direct summand isomorphic to R. Since Q is non-zero matrix factorization, there exists $j \in \mathbb{Z}_d$ such that $\operatorname{cok} Q_j \neq 0$. Thus, $\operatorname{cok} Q_j$ has a direct summand isomorphic to R. By Proposition 2.3.1, this implies that Q has a direct summand isomorphic to $\mathcal{P}_j$. Since Q is indecomposable, we have that $Q \cong \mathcal{P}_j$. The statement about indecomposable injectives follows immediately because of Lemma 2.2.13.

To show (ii), let $Q = (Q_1, Q_2, \ldots, Q_d) \in \mathrm{MF}_S^d(f)$ be an arbitrary projective of size n. Then, since Q is projective, there is an isomorphism of matrix factorizations $\bigoplus_{i \in \mathbb{Z}_d} \mathcal{P}_i^n \cong P(Q) \cong Q \oplus \Omega_{\mathrm{MF}_S^d(f)}(Q)$. It follows that $\operatorname{cok} Q_i \cong R^{s_i}$ for some $0 \leq s_i \leq n$, where $s_i = 0$ means $\operatorname{cok} Q_i = 0$. Using Proposition 2.3.1, we have that $Q \cong \bigoplus \mathcal{P}_i^{s_i}$ as desired.

The third statement follows by combining (i) and Proposition 2.3.1. □

In Chapter 3.1 we will see that if the regular ring S is complete, then the Krull-Remak-Schmidt Theorem holds in $\mathrm{MF}_S^d(f)$. However, for an arbitrary regular local ring, Proposition 2.3.1 and Corollary 2.3.2 combine to give us "cancellation" of projective objects without the full strength of the Krull-Remak-Schmidt Theorem.

Proposition 2.3.3. *Let $X, X' \in \mathrm{MF}_S^d(f)$ and let $Q \in \mathrm{MF}_S^d(f)$ be projective. If $X \oplus Q \cong X' \oplus Q$, then $X \cong X'$ in $\mathrm{MF}_S^d(f)$.*

Proof. First assume that Q is an indecomposable projective, that is, $Q = \mathcal{P}_i$ for $i \in \mathbb{Z}_d$. We claim that that $\mathcal{P}_i \in \mathrm{MF}_S^d(f)$ has a local endomorphism ring. To see this, notice that if $(\alpha_1, \ldots, \alpha_d) \in \mathrm{End}_{\mathrm{MF}_S^d(f)}(\mathcal{P}_i)$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_d$. Then the map which sends $(\alpha_1, \alpha_1, \ldots, \alpha_1) \mapsto \alpha_1 \in S$ forms an isomorphism $\mathrm{End}_{\mathrm{MF}_S^d(f)}(\mathcal{P}_i) \cong S$. Since idempotents split in $\mathrm{MF}_S^d(f)$ (see A.1), we may apply [LW12, Lemma 1.2] to conclude that $\mathcal{P}_i$ can be cancelled. In other words, we have shown that $X \oplus \mathcal{P}_i \cong X' \oplus \mathcal{P}_i$ implies that $X \cong X'$.

Finally, applying 2.3.2(ii) for arbitrary projective Q, we may cancel one indecomposable projective at a time and conclude that $X \oplus Q \cong X' \oplus Q$ implies that $X \cong X'$ as required. □

The exact structure on $\mathrm{MF}_S^d(f)$ ensures that $\Omega_{MF_S^d(f)}(X)$ is *stably equivalent* to any matrix factorization K such that there exists a short exact sequence

$$K \rightarrowtail P \twoheadrightarrow X$$

where P is projective. This follows from the appropriate version of Schanuel's Lemma in $\mathrm{MF}^d_S(f)$.

Lemma 2.3.4. *(Schanuel's Lemma) Let $X \in \mathrm{MF}^d_S(f)$ and suppose*

$$K \overset{q}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} X \qquad \text{and} \qquad K' \overset{q'}{\rightarrowtail} P' \overset{p'}{\twoheadrightarrow} X$$

are short exact sequences of matrix factorizations with P and P' projective. Then $P \oplus K' \cong K \oplus P'$. □

We omit the proof as it follows from [Büh10, Proposition 2.12]. The next Lemma follows directly from Lemma 2.3.4, Proposition 2.3.3, and the sequences (2.2.4) and (2.2.6).

Lemma 2.3.5. *Let $X, X' \in \mathrm{MF}^d_S(f)$. Then*

(i) $\Omega_{\mathrm{MF}^d_S(f)}(X \oplus X') \cong \Omega_{\mathrm{MF}^d_S(f)}(X) \oplus \Omega_{\mathrm{MF}^d_S(f)}(X')$

(ii) $\Omega^-_{\mathrm{MF}^d_S(f)}(X \oplus X') \cong \Omega^-_{\mathrm{MF}^d_S(f)}(X) \oplus \Omega^-_{\mathrm{MF}^d_S(f)}(X')$

(iii) $\Omega_{\mathrm{MF}^d_S(f)}(X)$ *(respectively $\Omega^-_{\mathrm{MF}^d_S(f)}(X)$) is projective if and only if X is projective.*

□

As a consequence, both $\Omega_{\mathrm{MF}^d_S(f)}(-)$ and $\Omega^-_{\mathrm{MF}^d_S(f)}(-)$ define additive functors from the stable category $\underline{\mathrm{MF}}^d_S(f)$ to itself.

Proposition 2.3.6. *Let $X \in \mathrm{MF}^d_S(f)$ be of size n and $\mathcal{P} = \bigoplus_{i=1}^d \mathcal{P}_i$. Then we have isomorphisms*

(i) $\Omega_{\mathrm{MF}^d_S(f)}(\Omega^-_{\mathrm{MF}^d_S(f)}(X)) \cong X \oplus \mathcal{P}^{(d-2)n}$,

(ii) $\Omega^-_{\mathrm{MF}^d_S(f)}(\Omega_{\mathrm{MF}^d_S(f)}(X)) \cong X \oplus \mathcal{P}^{(d-2)n}$, *and*

(iii) $\Omega_{\mathrm{MF}^d_S(f)}(X) \cong \Omega^-_{\mathrm{MF}^d_S(f)}(X)$.

Proof. Since $\Omega^-_{\mathrm{MF}^d_S(f)}(X)$ is of size $(d-1)n$, there is a short exact sequence

$$\Omega_{\mathrm{MF}^d_S(f)}(\Omega^-_{\mathrm{MF}^d_S(f)}(X)) \rightarrowtail \mathcal{P}^{(d-1)n} \twoheadrightarrow \Omega^-_{\mathrm{MF}^d_S(f)}(X).$$

By applying Schanuel's lemma to this sequence and the sequence (2.2.6), we find that

$$\Omega_{\mathrm{MF}^d_S(f)}(\Omega^-_{\mathrm{MF}^d_S(f)}(X)) \oplus \mathcal{P}^n \cong X \oplus \mathcal{P}^{(d-1)n}.$$

We may cancel one copy of $\mathcal{P}^n$ from both sides by Proposition 2.3.3 to obtain the first statement. Dually, the second statement follows from the injective version of Schanuel's Lemma.

In order to prove (iii), we construct an explicit isomorphism. For each $k \in \mathbb{Z}_d$ define an S-homomorphism $\alpha_k : \widehat{F}_k \to \widehat{F}_k$ by

$$\alpha_k = \begin{pmatrix} 1_{F_{k+1}} & \theta^X_{(k+1)(k+2)} & \theta^X_{(k+1)(k+3)} & \cdots & \theta^X_{(k+1)(k-1)} \\ 0 & 1_{F_{k+2}} & \theta^X_{(k+2)(k+3)} & \cdots & \theta^X_{(k+2)(k-1)} \\ 0 & 0 & 1_{F_{k+3}} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \theta^X_{(k-2)(k-1)} \\ 0 & 0 & \cdots & & 1_{F_{k-1}} \end{pmatrix}$$

where $\theta^X_{ji} = \varphi_j \varphi_{j+1} \cdots \varphi_{i-2}\varphi_{i-1}$ for $i \neq j$ and the identity on F_j for $i = j$. Notice that each α_k is an isomorphism and, by using the explicit descriptions of Ω_k and Ω^-_k in (2.2.10), we have that the diagram

$$\begin{array}{ccc} \widehat{F}_{k+1} & \xrightarrow{\Omega_k} & \widehat{F}_k \\ \downarrow{\scriptstyle \alpha_{k+1}} & & \downarrow{\scriptstyle \alpha_k} \\ \widehat{F}_{k+1} & \xrightarrow{\Omega^-_k} & \widehat{F}_k \end{array}$$

commutes for all $k \in \mathbb{Z}_d$. Hence, we have an isomorphism of matrix factorizations $(\alpha_1, \ldots, \alpha_d)$: $\Omega_{\mathrm{MF}^d_S(f)}(X) \to \Omega^-_{\mathrm{MF}^d_S(f)}(X)$. $\square$

Remark 2.3.7. In the case $d = 2$, no projective summands occur in the first two isomorphisms of Proposition 2.3.6. This agrees with what we saw in Example 2.2.11 which said that the syzygy and cosyzygy operations are isomorphic to the shift functor:

$$\Omega_{\mathrm{MF}^2_S(f)}(\varphi, \psi) \cong (\psi, \varphi) \cong \Omega^-_{\mathrm{MF}^2_S(f)}(\varphi, \psi)$$

for any $(\varphi, \psi) \in \mathrm{MF}^2_S(f)$. From this we can see that all three statements of Proposition 2.3.6 are immediate when $d = 2$. In particular, the isomorphism in Proposition 2.3.6 (iii) is just the identity. In contrast, the isomorphism constructed in Proposition 2.3.6 (iii) when $d = 3$ is

$$\begin{array}{ccccccc}
F_2 \oplus F_3 & \xrightarrow{\begin{pmatrix} -\varphi_1 & -\varphi_1\varphi_2 \\ 1_{F_2} & 0 \end{pmatrix}} & F_1 \oplus F_2 & \xrightarrow{\begin{pmatrix} -\varphi_3 & -\varphi_3\varphi_1 \\ 1_{F_1} & 0 \end{pmatrix}} & F_3 \oplus F_1 & \xrightarrow{\begin{pmatrix} -\varphi_2 & -\varphi_2\varphi_3 \\ 1_{F_3} & 0 \end{pmatrix}} & F_2 \oplus F_3 \\
\begin{pmatrix} 1_{F_2} & \varphi_2 \\ 0 & 1_{F_3} \end{pmatrix}\Big\downarrow & & \begin{pmatrix} 1_{F_1} & \varphi_1 \\ 0 & 1_{F_2} \end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix} 1_{F_3} & \varphi_3 \\ 0 & 1_{F_1} \end{pmatrix} & & \Big\downarrow\begin{pmatrix} 1_{F_2} & \varphi_2 \\ 0 & 1_{F_3} \end{pmatrix} \\
F_2 \oplus F_3 & \xrightarrow[\begin{pmatrix} 0 & -\varphi_1\varphi_2 \\ 1_{F_2} & -\varphi_2 \end{pmatrix}]{} & F_1 \oplus F_2 & \xrightarrow[\begin{pmatrix} 0 & -\varphi_3\varphi_1 \\ 1_{F_1} & -\varphi_1 \end{pmatrix}]{} & F_3 \oplus F_1 & \xrightarrow[\begin{pmatrix} 0 & -\varphi_2\varphi_3 \\ 1_{F_3} & -\varphi_3 \end{pmatrix}]{} & F_2 \oplus F_3
\end{array}.$$

The next Proposition uses Proposition 2.3.6 to show that each object in $\mathrm{MF}^d_S(f)$ has a projective resolution which is periodic of period at most 2. A projective resolution in this context is with respect to the exact structure on $\mathrm{MF}^d_S(f)$ (see [Büh10, p. 12.1]).

Proposition 2.3.8. *Let $X \in \mathrm{MF}^d_S(f)$. Then X has a projective resolution which is periodic* *with period at most 2:*

$$\cdots \xrightarrow{q} I(X) \xrightarrow{p} P(X) \xrightarrow{q} I(X) \xrightarrow{p} P(X) \twoheadrightarrow X.$$

Proof. Set $\Omega(X) = \Omega_{\mathrm{MF}^d_S(f)}(X)$ and $\Omega^-(X) = \Omega^-_{\mathrm{MF}^d_S(f)}(X)$. Let $\alpha : \Omega(X) \to \Omega^-(X)$ be the isomorphism constructed in Proposition 2.3.6. Then we have two diagrams

$$\begin{array}{ccccccccc} I(X) & & \longrightarrow & & P(X) & & \longrightarrow & & I(X) \\ & \searrow^{\alpha^{-1}\eta^X} & & \nearrow_{\epsilon^X} & & \searrow^{\rho^X} & & \nearrow_{\lambda^X} & \\ & & \Omega(X) & & & & X & & \end{array} \tag{2.3.1}$$

and

$$\begin{array}{ccccccccc} P(X) & & \longrightarrow & & I(X) & & \longrightarrow & & P(X) \\ & \searrow^{\rho^X} & & \nearrow_{\lambda^X} & & \searrow^{\alpha^{-1}\eta^X} & & \nearrow_{\epsilon^X} & \\ & & X & & & & \Omega(X) & & \end{array} . \tag{2.3.2}$$

The middle sequence in (2.3.2) is short exact since we have a commutative diagram

$$\begin{array}{ccccc} X & \overset{\lambda^X}{\rightarrowtail} & I(X) & \overset{\alpha^{-1}\eta^X}{\twoheadrightarrow} & \Omega(X) \\ \| & & \| & & \downarrow{\scriptstyle \alpha} \\ X & \overset{\lambda^X}{\rightarrowtail} & I(X) & \overset{\eta^X}{\twoheadrightarrow} & \Omega^-(X) \end{array}$$

with vertical isomorphisms. The desired resolution follows by splicing together (2.3.1) and (2.3.2), that is, by setting $p = \epsilon^X \alpha^{-1} \eta^X$ and $q = \lambda^X \rho^X$. □

Proposition 2.3.1 and Corollary 2.3.2 give us a clearer picture of the structure of matrix factorizations and the MCM R-modules they encode.

Proposition 2.3.9. *Let* $X = (\varphi_1, \dots, \varphi_d) \in \mathrm{MF}^d_S(f)$. *Then*

$$X \cong \tilde{X} \oplus \mathcal{P}_1^{s_1} \oplus \mathcal{P}_2^{s_2} \oplus \cdots \oplus \mathcal{P}_d^{s_d} \tag{2.3.3}$$

for some stable matrix factorization $\tilde{X} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_d)$ *and integers* $s_k \geq 0$, $k \in \mathbb{Z}_d$. *The* *integers s_k are uniquely determined and $\tilde{X}$ is unique up to isomorphism.*

Proof. For each $k \in \mathbb{Z}_d$, we may write $\operatorname{cok} \varphi_k \cong M_k \oplus R^{s_k}$ for a stable MCM R-module M_k. The integer s_k is uniquely determined by $\operatorname{cok} \varphi_k$. By Proposition 2.3.1, X has a direct summand isomorphic to $\mathcal{P}_k^{s_k}$. Hence, we may write $X \cong \tilde{X} \oplus \left(\bigoplus_{k \in \mathbb{Z}_d} \mathcal{P}_k^{s_k}\right)$ for some $\tilde{X} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_d) \in \mathrm{MF}_S^d(f)$. By construction, $\operatorname{cok} \tilde{\varphi}_k$ has no free summands for each $k \in \mathbb{Z}_d$. Hence, $\tilde{X}$ is a stable matrix factorization and $\operatorname{cok} \tilde{\varphi}_k \cong M_k$ for each $k \in \mathbb{Z}_d$.

Suppose we have another decomposition $X \cong \tilde{Y} \oplus \left(\bigoplus_{k \in \mathbb{Z}_d} \mathcal{P}_k^{s_k}\right)$. By Proposition 2.3.3, we may cancel the indecomposable projectives and conclude that $\tilde{X} \cong \tilde{Y}$ as desired. □

Corollary 2.3.10. *Let $X = (\varphi_1, \ldots, \varphi_d) \in \mathrm{MF}_S^d(f)$ of size n. Then*

$$\Omega_{\mathrm{MF}_S^d(f)}(X) \cong \tilde{\Omega} \oplus \mathcal{P}_1^{m_1} \oplus \cdots \oplus \mathcal{P}_d^{m_d} \tag{2.3.4}$$

where $m_k = n - \mu_R(\operatorname{cok} \varphi_k)$ and $\tilde{\Omega} \in \mathrm{MF}_S^d(f)$ is stable. Furthermore, $\tilde{\Omega}$ is of size $\sum_{k=1}^d \mu_R(\operatorname{cok} \varphi_k) - n$.

Proof. The isomorphism (2.3.4) follows by combining Propositions 2.3.9 and 2.2.12. Let $\ell \geq 0$ be the size of $\tilde{\Omega}$. Since $\Omega_{\mathrm{MF}_S^d(f)}(X)$ is of size $(d-1)n$, we have that

$$(d-1)n = \ell + \sum_{k=1}^{d} m_k = \ell + dn - \sum_{k=1}^{d} \mu_R(\operatorname{cok} \varphi_k).$$

Thus, $\ell = \sum_{k=1}^d \mu_R(\operatorname{cok} \varphi_k) - n$. □

Over the hypersurface ring $R = S/(f)$, the reduced syzygy of an indecomposable non-free MCM R-module is again indecomposable. This is a special case of a theorem of Herzog (see [LW12, Lemma 9.14]). Proposition 2.3.11 gives an analogous result for matrix factorizations.

Proposition 2.3.11. *Let $X \in \mathrm{MF}_S^d(f)$ be a indecomposable non-projective matrix factorization and let $\Omega_{\mathrm{MF}_S^d(f)}(X) \cong \tilde{\Omega} \oplus P$ be a decomposition of the form (2.3.4). Then $\tilde{\Omega}$ is indecomposable.*

Proof. First, note that $\tilde{\Omega} \neq 0$. Indeed, if $\tilde{\Omega}$ was zero, then $\Omega_{\mathrm{MF}_S^d(f)}(X)$ would be projective and Proposition 2.3.5 would imply that X is projective as well, which is not the case. So, assume $\tilde{\Omega} = Y_1 \oplus Y_2$ for some non-zero $Y_1, Y_2 \in \mathrm{MF}_S^d(f)$. Since $\tilde{\Omega}$ is stable, the direct summands Y_1 and Y_2 are also stable. Then

$$\begin{aligned}\Omega^-_{\mathrm{MF}_S^d(f)}(\Omega_{\mathrm{MF}_S^d(f)}(X)) &\cong \Omega^-_{\mathrm{MF}_S^d(f)}(\tilde{\Omega}) \oplus \Omega^-_{\mathrm{MF}_S^d(f)}(P) \\ &\cong \Omega^-_{\mathrm{MF}_S^d(f)}(Y_1) \oplus \Omega^-_{\mathrm{MF}_S^d(f)}(Y_2) \oplus \Omega^-_{\mathrm{MF}_S^d(f)}(P).\end{aligned}$$

For $i = 1, 2$, decompose $\Omega^-_{\mathrm{MF}_S^d(f)}(Y_i) \cong U_i \oplus P_i$, for some stable U_i and projective P_i. Now, applying Proposition 2.3.6, we have that

$$X \oplus \mathcal{P}^{(d-2)n} \cong U_1 \oplus U_2 \oplus P_1 \oplus P_2 \oplus \Omega^-_{\mathrm{MF}_S^d(f)}(P)$$

where n is the size of X. Since both sides of this isomorphism are decomposed into the form (2.3.3), we have that $U_1 \oplus U_2 \cong X$. But X is indecomposable, so one of U_1 or U_2 must be zero. Re-indexing if necessary, we may assume $U_1 = 0$. This implies that $\Omega^-_{\mathrm{MF}_S^d(f)}(Y_1)$ is projective and therefore Y_1 is projective. However, this is a contradiction since Y_1 is a non-zero stable matrix factorization. Hence, $\tilde{\Omega}$ is indecomposable. □

So far, we have refrained from assuming that X is a reduced matrix factorization. On the other hand, if we do assume that $X \in \mathrm{MF}_S^d(f)$ is reduced, we obtain slightly stronger versions of 2.2.12, 2.3.9, 2.3.10, and 2.3.11.

Corollary 2.3.12. *Let $X \in \mathrm{MF}_S^d(f)$ be reduced. Then the following hold.*

(i) $\operatorname{cok} \Omega_k \cong \operatorname{syz}^1_R(\operatorname{cok} \varphi_k)$ *for each* $k \in \mathbb{Z}_d$.

(ii) Both X *and* $\Omega_{\mathrm{MF}_S^d(f)}(X)$ *are stable.*

(iii) If X *is indecomposable, then* $\Omega_{\mathrm{MF}_S^d(f)}(X)$ *is indecomposable.*

Proof. By Corollary 1.2.8(ii), there is a one-to-one correspondence between reduced matrix factorizations in $\mathrm{MF}_S^2(f)$ and stable MCM R-modules. If $X = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in \mathrm{MF}_S^d(f)$ is reduced, then $(\varphi_k, \varphi_{k+1}\varphi_{k+2}\cdots\varphi_{k-1})$ is a reduced matrix factorization in $\mathrm{MF}_S^2(f)$ for each $k \in \mathbb{Z}_d$. Hence, $\operatorname{cok}\varphi_k$ is a stable MCM R-module for each $k \in \mathbb{Z}_d$ and $\operatorname{cok}(\varphi_{k+1}\varphi_{k+2}\cdots\varphi_{k-1})$ is its reduced first syzygy. Since, by Proposition 2.2.12, $\operatorname{cok}\Omega_k \cong \operatorname{cok}(\varphi_{k+1}\varphi_{k+2}\cdots\varphi_{k-1})$, the first statement follows. The second statement follows from Proposition 2.3.1 and the third follows from the second and Proposition 2.3.11. $\square$

3 | Endomorphisms of the Projective Generator

Let $(S, \mathfrak{n}, \mathbf{k})$ be a regular local ring, f a non-zero non-unit in S, and $d \geq 2$ an integer. In this chapter, we will show that $\mathrm{MF}_S^d(f)$ is equivalent to the category of MCM modules over a certain non-commutative S-algebra which is finitely generated and free as an S-module. This extends a result of Solberg [Sol89, Proposition 1.3] for all $d \geq 2$. As a consequence we will conclude that, if S is complete, the Krull-Remak-Schmidt Theorem (KRS) holds in $\mathrm{MF}_S^d(f)$.

3.1 $\mathrm{MF}_S^d(f)$ as a category of modules

Recall from Chapter 2 the projective (equivalently injective) matrix factorizations $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_d$, and their direct sum $\mathcal{P} = \bigoplus_{i \in \mathbb{Z}_d} \mathcal{P}_i$. Our first step is to understand the morphisms from $\mathcal{P}_j$ to $\mathcal{P}_i$ for any $i, j \in \mathbb{Z}_d$.

Definition 3.1.1. For $i \neq j \in \mathbb{Z}_d$, let $e_{ij} \in \mathrm{Hom}_S(S, S)^d$ denote the d-tuple of homomorphisms such that the $j+1, j+2, \ldots, i-1, i$ components are multiplication by f while the rest are the identity on S. For each $i \in \mathbb{Z}_d$, let $e_{ii} = 1_{\mathcal{P}_i}$, the identity on $\mathcal{P}_i$.

For instance, if $i \in \mathbb{Z}_d$, the i-th map in $e_{i(i-1)}$ is multiplication by f and the rest are the identity on S while $e_{(i-1)i}$ is of the form $(f, f, \ldots, f, 1, f, \ldots, f, f)$, where only the i-th component is the identity on S.

Lemma 3.1.2. *For all $i, j \in \mathbb{Z}_d$, $\mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}_j, \mathcal{P}_i) = S \cdot e_{ij}$. In particular, the morphisms from $\mathcal{P}_j$ to $\mathcal{P}_i$ form a free S-module of rank 1.*

Proof. The morphisms from $\mathcal{P}_j$ to $\mathcal{P}_i$ are tuples of the form $(\alpha_1, \ldots, \alpha_d)$ for some $\alpha_k \in S$. If $(\alpha_1, \ldots, \alpha_d) \in \operatorname{Hom}_{\mathrm{MF}^d_S(f)}(\mathcal{P}_i, \mathcal{P}_i)$, then it is easy to see that $\alpha_1 = \alpha_2 = \cdots = \alpha_d$ and so $\operatorname{Hom}_{\mathrm{MF}^d_S(f)}(\mathcal{P}_i, \mathcal{P}_i) = S \cdot e_{ii}$. Suppose $(\alpha_1, \ldots, \alpha_d) \in \operatorname{Hom}_{\mathrm{MF}^d_S(f)}(\mathcal{P}_j, \mathcal{P}_i)$ where $i \neq j$. Consider the following diagram, which commutes since $(\alpha_1, \ldots, \alpha_d)$ is a morphism of matrix factorizations:

$$\begin{array}{ccccccccccccc}
\mathcal{P}_j & \quad & \cdots & \xrightarrow{1} & S & \xrightarrow{1} & S & \xrightarrow{1} & \cdots & \xrightarrow{1} & S & \xrightarrow{f} & S & \xrightarrow{1} & \cdots \\
\downarrow & & & & \downarrow{\scriptstyle \alpha_{i+1}} & & \downarrow{\scriptstyle \alpha_i} & & & & \downarrow{\scriptstyle \alpha_{j+1}} & & \downarrow{\scriptstyle \alpha_j} & & \\
\mathcal{P}_i & \quad & \cdots & \xrightarrow{1} & S & \xrightarrow{f} & S & \xrightarrow{1} & \cdots & \xrightarrow{1} & S & \xrightarrow{1} & S & \xrightarrow{1} & \cdots
\end{array}$$

We conclude that $\alpha_{j+1} = \alpha_{j+2} = \cdots = \alpha_i$, $\alpha_j = \alpha_{j-1} = \cdots = \alpha_{i+1}$, and $\alpha_i = f\alpha_{i+1}$. Thus, each component of the morphism can be rewritten in terms of the element $\alpha_{i+1} \in S$. It follows that $(\alpha_1, \ldots, \alpha_d) = \alpha_{i+1} e_{ij}$, that is, $\operatorname{Hom}_{\mathrm{MF}^d_S(f)}(\mathcal{P}_j, \mathcal{P}_i)$ is generated by e_{ij} as an S-module. Since the components of e_{ij} are given by multiplication by non-zero elements of S, a morphism $s \cdot e_{ij} = 0$ if and only if $s = 0$. Hence, $\operatorname{Hom}_{\mathrm{MF}^d_S(f)}(\mathcal{P}_j, \mathcal{P}_i)$ is in fact a free S-module of rank 1 for all $i, j \in \mathbb{Z}_d$. □

Let $\Gamma = \operatorname{End}_{\mathrm{MF}^d_S(f)}(\mathcal{P})^{\mathrm{op}}$ where $\mathcal{P} = \bigoplus_{i=1}^d \mathcal{P}_i$. As S-modules, $\Gamma \cong \bigoplus_{i,j \in \mathbb{Z}_d} \operatorname{Hom}(\mathcal{P}_j, \mathcal{P}_i)$ and therefore, Lemma 3.1.2 implies that Γ is a free S-module of rank d^2. For each pair $i, j \in \mathbb{Z}_d$, use the same symbol e_{ij} to denote the image of the generator of $\operatorname{Hom}(\mathcal{P}_j, \mathcal{P}_i)$ under the natural inclusion $\operatorname{Hom}(\mathcal{P}_j, \mathcal{P}_i) \to \Gamma$. The set $\{e_{ij}\}_{i,j \in \mathbb{Z}_d}$ forms a basis for Γ as an S-module. We record the basic rules for multiplication of the basis elements e_{ij}.

Lemma 3.1.3. *The basis elements $\{e_{ij}\}_{i,j \in \mathbb{Z}_d}$ satisfy the following properties.*

(i) $e_{ij} e_{pq} \neq 0$ *if and only if* $i = q$

(ii) $e_{ii}^2 = e_{ii}$ *for all* $i \in \mathbb{Z}_d$

(iii) $\sum_{i=1}^d e_{ii} = 1_\Gamma$

(iv) $e_{ij} e_{ii} = e_{ij}$ *and* $e_{jj} e_{ij} = e_{ij}$ *for all* $i, j \in \mathbb{Z}_d$

(v) $e_{i(i-1)}e_{ji} = \begin{cases} fe_{(i-1)(i-1)} & \textit{if } j = i-1 \\ e_{j(i-1)} & \textit{otherwise} \end{cases}$

(vi) $e_{ij}e_{(i+1)i} = \begin{cases} fe_{(i+1)(i+1)} & j = i+1 \\ e_{(i+1)j} & \textit{otherwise} \end{cases}$

(vii) $\left(\sum_{i=1}^{d} e_{i(i-1)}\right) e_{jj} = e_{j(j-1)} = e_{(j-1)(j-1)} \left(\sum_{i=1}^{d} e_{i(i-1)}\right)$ *for all* $j \in \mathbb{Z}_d$

□

Lemma 3.1.4. *Let* $i, j \in \mathbb{Z}_d$ *with* $i \neq j$*. Then* e_{ij} *can be written as a product of basis elements of the form* $e_{\ell(\ell-1)}$*. In particular,* $e_{ij} = e_{(j+1)j}e_{(j+2)(j+1)} \cdots e_{(i-1)(i-2)}e_{i(i-1)}$*.*

Proof. Let $i \neq j \in \mathbb{Z}_d$. Lemma 3.1.3 (vi) implies that, for any $\ell \neq j \in \mathbb{Z}_d$, the element $e_{\ell(\ell-1)}$ can be factored out of $e_{\ell j}$ on the right

$$e_{\ell j} = e_{(\ell-1)j}e_{\ell(\ell-1)}.$$

Since $i \neq j$, we may apply this equality for $\ell = i, i-1, \ldots, j+2, j+1$ which gives us the factorization

$$\begin{aligned} e_{ij} &= e_{(i-1)j}e_{i(i-1)} \\ &= e_{(i-2)j}e_{(i-1)(i-2)}e_{i(i-1)} \\ &= \cdots \\ &= e_{(j+1)j}e_{(j+2)(j+1)} \cdots e_{(i-1)(i-2)}e_{i(i-1)}. \end{aligned}$$

□

The element $\sum_{i=1}^{d} e_{i(i-1)} \in \Gamma$ is of particular interest because of the following.

Lemma 3.1.5. *Let $z = \sum_{i=1}^{d} e_{i(i-1)}$ and $s \geq 1$ an integer. Write $s = dq + r$ for $q \geq 0$ and $0 \leq r < d$. Then*

$$z^s = f^q \sum_{i=1}^{d} e_{i(i-r)}.$$

In particular, $z^d = f \cdot 1_\Gamma$.

Proof. If $s = 1$ there is nothing to prove. Assume the formula holds for $s \geq 1$ and consider z^{s+1}. By induction,

$$z^{s+1} = z \cdot f^q \sum_{i=1}^{d} e_{i(i-r)}$$

where $s = dq + r$, $q \geq 0$, and $0 \leq r < d$. If $r = d - 1$, then, by Lemma 3.1.3,

$$z \cdot \sum_{i=1}^{d} e_{i(i-r)} = \sum_{i=1}^{d} e_{(i+1)i} e_{i(i+1)} = f \cdot 1_\Gamma.$$

Since $s = dq + d - 1$, we have that $s + 1 = d(q + 1)$. Hence,

$$z^{s+1} = f^{q+1} \sum_{i=1}^{d} e_{ii}$$

as needed. If $0 \leq r < d - 1$, then

$$z \cdot \sum_{i=1}^{d} e_{i(i-r)} = \sum_{i=1}^{d} e_{i(i-r-1)}$$

also by Lemma 3.1.3. In this case,

$$z^{s+1} = f^q \sum_{i=1}^{d} e_{i(i-(r+1))}$$

which completes the induction since $s + 1 = dq + (r + 1)$ with $0 \leq r + 1 < d$.

□

Let MCM(Γ) denote the full subcategory of finitely generated left Γ-modules which are free when viewed as S-modules via the inclusion $S \cdot 1_\Gamma \subset \Gamma$.

Theorem 3.1.6. *The categories* $\mathrm{MCM}(\Gamma)$ *and* $\mathrm{MF}^d_S(f)$ *are equivalent.*

Before the proof of Theorem 3.1.6, we record one consequence which will be of use in future sections.

Recall that the Krull-Remak-Schmidt Theorem (KRS) holds in an additive category if each object decomposes into a finite direct sum of indecomposable objects and if this direct sum decomposition is unique up to isomorphism and permutation of the indecomposable summands.

Corollary 3.1.7. *Assume that* S *is complete. Then KRS holds in* $\mathrm{MF}^d_S(f)$*.*

Proof. If S is complete, it is known that KRS holds in the category $\mathrm{MCM}(\Gamma)$ (for example see [Aus86, Section 1]). Hence, by Theorem 3.1.6, KRS also holds in $\mathrm{MF}^d_S(f)$. □

The rest of this section is dedicated to proving Theorem 3.1.6. We start by defining a functor $\mathcal{H} : \mathrm{MCM}(\Gamma) \to \mathrm{MF}^d_S(f)$ using the element $z = \sum_{i=1}^d e_{i(i-1)} \in \Gamma$. Let M be a Γ-module in $\mathrm{MCM}(\Gamma)$. Lemma 3.1.3 (i)-(iii) show that $e_{11}, \ldots, e_{dd}$ are orthogonal idempotents such that $e_{11} + e_{22} + \cdots + e_{dd} = 1_\Gamma$. Thus, M decomposes, as an S-module, into

$$M = e_{11}M \oplus \cdots \oplus e_{dd}M.$$

Since $M \in \mathrm{MCM}(\Gamma)$, each summand $e_{ii}M$ is a free S-module. Lemma 3.1.3 (vii) shows that left multiplication by $z \in \Gamma$ defines an S-homomorphism between free S-modules $z : e_{ii}M \to e_{(i-1)(i-1)}M$ for all $i \in \mathbb{Z}_d$.

Proposition 3.1.8. *Let* $M \in \mathrm{MCM}(\Gamma)$*. The* d*-tuple of* S*-homomorphisms*

$$(z : e_{22}M \to e_{11}M, z : e_{33}M \to e_{22}M \ldots, z : e_{11}M \to e_{dd}M)$$

forms a matrix factorization of f *in* $\mathrm{MF}^d_S(f)$*, where each map is multiplication by* $z = \sum_{i=1}^d e_{i(i-1)} \in \Gamma$.

Proof. In light of Lemma 3.1.5, the only piece that needs justification is that each of the free S-modules involved are of the same rank. To see this, let $i \in \mathbb{Z}_d$. The composition

$$e_{ii}M \xrightarrow{z^{d-1}} e_{(i+1)(i+1)}M \xrightarrow{z} e_{ii}M$$

is f times the identity on $e_{ii}M$. Similarly, the composition

$$e_{(i+1)(i+1)}M \xrightarrow{z} e_{ii}M \xrightarrow{z^{d-1}} e_{(i+1)(i+1)}M$$

is f times the identity on $e_{(i+1)(i+1)}M$. Since $e_{ii}M$ and $e_{(i+1)(i+1)}M$ are free over S, Lemma 1.2.9 implies that $\operatorname{rank}_S(e_{ii}M) = \operatorname{rank}_S(e_{(i+1)(i+1)}M)$. □

Following Proposition 3.1.8, the functor $\mathcal{H} : \mathrm{MCM}(\Gamma) \to \mathrm{MF}_S^d(f)$ is defined as follows: $\mathcal{H}(M) = (z : e_{22}M \to e_{11}M, \ldots, z : e_{11}M \to e_{dd}M)$ for any $M \in \mathrm{MCM}(\Gamma)$ and, for a homomorphism $h : M \to N$ in $\mathrm{MCM}(\Gamma)$, define $\mathcal{H}(h) = (h|_{e_{11}M}, \ldots, h|_{e_{dd}M})$, where $h|_{e_{ii}M}$ denotes the restriction of h to the S-direct summand $e_{ii}M$. Since h is a Γ-homomorphism, $h|_{e_{ii}M}$ maps $e_{ii}M$ into $e_{ii}N$. Since multiplication by an element of Γ commutes with any Γ-homomorphism, this d-tuple forms a morphism between the matrix factorizations $\mathcal{H}(M) \to \mathcal{H}(N)$. At this point we can prove that $\mathcal{H}$ is both full and faithful.

Proposition 3.1.9. *The functor* $\mathcal{H} : \mathrm{MCM}(\Gamma) \to \mathrm{MF}_S^d(f)$ *is full and faithful.*

Proof. Let $M, N \in \mathrm{MCM}(\Gamma)$. If $\mathcal{H}(h) = 0$ for some Γ-homomorphism $h : M \to N$, then $h|_{e_{ii}M} = 0$ for each $i \in \mathbb{Z}_d$. But this means that $h = \bigoplus_{i \in \mathbb{Z}_d} h|_{e_{ii}M} = 0$ implying that $\mathcal{H}$ is faithful.

In order to show that $\mathcal{H}$ is full, let $(\alpha_1, \ldots, \alpha_d) : \mathcal{H}(M) \to \mathcal{H}(N)$ be a morphism of matrix factorizations. So, $\alpha_i : e_{ii}M \to e_{ii}N$ and we have a commutative diagram

$$\begin{array}{ccc} e_{ii}M & \xrightarrow{z} & e_{(i-1)(i-1)}M \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i-1}} \\ e_{ii}N & \xrightarrow{z} & e_{(i-1)(i-1)}N \end{array} \tag{3.1.1}$$

for each $i \in \mathbb{Z}_d$. Let $h = \bigoplus_{j=1}^{d} \alpha_j : M \to N$ be the S-homomorphism given by $h(m) = \alpha_1(e_{11}m) + \alpha_2(e_{22}m) + \cdots + \alpha_d(e_{dd}m)$ for all $m \in M$. We claim that h is in fact a Γ-homomorphism and furthermore, $\mathcal{H}(h) = (\alpha_1, \ldots, \alpha_d)$. The second claim follows from the first and the definition of $\mathcal{H}$ and so our aim is to show that h is a homomorphism of Γ-modules. Since h is an S-homomorphism and Γ is a finitely generated free S-module with basis $\{e_{ij}\}_{i,j \in \mathbb{Z}_d}$, we would be done if we showed that $e_{ij}h(m) = h(e_{ij}m)$ for all $i, j \in \mathbb{Z}_d$ and $m \in M$. By Lemma 3.1.4, it suffices to show that the elements of the form $e_{k(k-1)}$ pass through h since each e_{ij} is a product of elements of this form.

Let $i \in \mathbb{Z}_d$ and $m \in M$. By Lemma 3.1.3(vii), multiplication by z on $e_{ii}M$ (respectively on $e_{ii}N$) coincides with multiplication by the element $e_{i(i-1)} \in \Gamma$. Therefore, the diagram (3.1.1) implies that

$$\alpha_{i-1}(e_{i(i-1)}e_{ii}m) = e_{i(i-1)}\alpha_i(e_{ii}m).$$

Since $e_{i(i-1)}e_{ii}m \in e_{(i-1)(i-1)}M$, the term on the left hand side is precisely $h(e_{i(i-1)}m)$. On the other hand, since $e_{ii}h(m) = \alpha_i(e_{ii}m)$, we have that

$$\begin{aligned} e_{i(i-1)}h(m) &= e_{i(i-1)}e_{ii}h(m) \\ &= e_{i(i-1)}\alpha_i(e_{ii}m). \end{aligned}$$

Together, we have that $e_{i(i-1)}h(m) = h(e_{i(i-1)}m)$ as desired. Thus, h is a Γ-homomorphism and $\mathcal{H}$ is full. □

To show that $\mathcal{H}$ is dense, we define a functor $\mathcal{F} : \mathrm{MF}_S^d(f) \to \mathrm{MCM}(\Gamma)$ which is given by $\mathcal{F}(\square) = \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}, \square)$, where $\mathcal{P} = \bigoplus_{i=1}^{d} \mathcal{P}_i$. For a matrix factorization $X \in \mathrm{MF}_S^d(f)$, $\mathcal{F}(X)$ is a left Γ-module by pre-composing any morphism $\mathcal{P} \to X$ with an element of Γ. In order to show that the image of $\mathcal{F}$ does indeed land in $\mathrm{MCM}(\Gamma)$, we must show that $\mathcal{F}(X)$ is a free S-module. This requires an explicit description of the morphisms $\mathcal{P} \to X$.

Recall that $\mathcal{P}_i$ is the matrix factorization whose i-th component is multiplication by f on S while the rest are the identity on S. For $k \in \mathbb{Z}_d$, let $D_k : S^d \to S^d$ be given by

$D_k(a_1,\ldots,a_d) = (a_1,\ldots,a_{k-1}, fa_k, a_{k+1},\ldots,a_d)$ for any $(a_1,\ldots,a_d) \in S^d$. Then $\mathcal{P} = (D_1 : S^d \to S^d, D_2 : S^d \to S^d, \ldots, D_d : S^d \to S^d)$.

Lemma 3.1.10. *Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$ and let $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}, X) = \mathcal{F}(X)$. For each $k \in \mathbb{Z}_d$, we may write $\alpha_k = \begin{pmatrix} \alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{kd} \end{pmatrix}$ for some $\alpha_{ki} \in \mathrm{Hom}_S(S, F_k)$. Then, for any $j \neq 0 \in \mathbb{Z}_d$,*

$$\alpha_{k(k+j)} = \varphi_k \varphi_{k+1} \cdots \varphi_{k+j-1} \alpha_{(k+j)(k+j)}.$$

Proof. Similar to the proof of Lemma 2.2.15, the formula follows from the following commutative diagram:

$$\begin{array}{ccccccccc}
S^d & \xrightarrow{D_{k+j-1}} & S^d & \xrightarrow{D_{k+j-2}} & \cdots & \xrightarrow{D_{k+1}} & S^d & \xrightarrow{D_k} & S^d \\
\downarrow{\scriptstyle \alpha_{k+j}} & & \downarrow{\scriptstyle \alpha_{k+j-1}} & & & & \downarrow{\scriptstyle \alpha_{k+1}} & & \downarrow{\scriptstyle \alpha_k} \\
F_{k+j} & \xrightarrow{\varphi_{k+j-1}} & F_{k+j-1} & \xrightarrow{\varphi_{k+j-2}} & \cdots & \xrightarrow{\varphi_{k+1}} & F_{k+1} & \xrightarrow{\varphi_k} & F_k.
\end{array}$$

The commutativity of the outermost rectangle gives us that

$$\alpha_k D_k D_{k+1} \cdots D_{k+j-1} = \varphi_k \varphi_{k+1} \cdots \varphi_{k+j-1} \alpha_{k+j}.$$

Since $j \neq 0$, the composition $D_k D_{k+1} \cdots D_{k+j-1}$ is the identity on the $(k+j)$-th component of S^d. Therefore, if we compare the $(k+j)$-th components of the homomorphisms on either side of the above equality, we find that $\alpha_{k(k+j)} = \varphi_k \varphi_{k+1} \cdots \varphi_{k+j-1} \alpha_{(k+j)(k+j)}$ as desired. □

Let $X = (\varphi_1 : F_2 \to F_1, \cdots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$ be a matrix factorization. Recall the homomorphisms $\theta_{ki}^X : F_i \to F_k$, $i, k \in \mathbb{Z}_d$, which are given by

$$\theta_{ki}^X = \begin{cases} 1_{F_k} & i = k \\ \varphi_k \varphi_{k+1} \cdots \varphi_{i-2} \varphi_{i-1} & i \neq k. \end{cases}$$

For each $(g_1, \ldots, g_d) \in \bigoplus_{i=1}^{d} F_i$, we associate a d-tuple of S-homomorphisms

$$\theta^X(g_1, \ldots, g_d) \coloneqq \left(\begin{pmatrix} \theta_{k1}^X g_1 & \theta_{k2}^X g_2 & \cdots & \theta_{kd}^X g_d \end{pmatrix}\right)_{k=1}^{d}.$$

Here $\theta_{ki}^X g_i$ is being identified with its image in $\mathrm{Hom}_S(S, F_k)$ under the natural isomorphism $F_k \cong \mathrm{Hom}_S(S, F_k)$. When X is clear from context, we will omit the superscripts.

Lemma 3.1.11. *Let* $X = (\varphi_1 : F_2 \to F_1, \cdots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$*. Then* $\theta(g_1, \ldots, g_d) \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}, X)$ *for any* $(g_1, \ldots, g_d) \in \bigoplus_{i=1}^{d} F_i$*. Furthermore, the map* $\theta : \bigoplus_{i=1}^{d} F_i \to \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}, X)$ *is an isomorphism of* S*-modules.*

Proof. First, we show that $\theta(g_1, \ldots, g_d)$ as defined is in fact a morphism of matrix factorizations between $\mathcal{P}$ and X. What needs to be shown is the commutativity of the diagram

$$\begin{array}{ccc} S^d & \xrightarrow{D_k} & S^d \\ {\scriptstyle\begin{pmatrix} \theta_{(k+1)1} g_1 & \theta_{(k+1)2} g_2 & \cdots & \theta_{(k+1)d} g_d \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle\begin{pmatrix} \theta_{k1} g_1 & \theta_{k2} g_2 & \cdots & \theta_{kd} g_d \end{pmatrix}} \\ F_{k+1} & \xrightarrow{\varphi_k} & F_k \end{array}$$

for all $k \in \mathbb{Z}_d$. Notice that $\varphi_k \theta_{(k+1)k} = \varphi_k \varphi_{k+1} \varphi_{k+2} \cdots \varphi_{k-1} = f \cdot 1_{F_k} = f \theta_{kk}$ and $\varphi_k \theta_{(k+1)i} = \theta_{ki}$ for all $i \neq k$. Therefore, we have that

$$\begin{aligned} \varphi_k \begin{pmatrix} \theta_{(k+1)1} g_1 & \cdots & \theta_{(k+1)d} g_d \end{pmatrix} &= \begin{pmatrix} \theta_{k1} g_1 & \cdots & f\theta_{kk} g_k & \cdots & \theta_{kd} g_d \end{pmatrix} \\ &= \begin{pmatrix} \theta_{k1} g_1 & \cdots & \theta_{kk} g_k & \cdots & \theta_{kd} g_d \end{pmatrix} D_k \end{aligned}$$

which implies the commutativity of the diagram as desired.

In order to show θ is an S-module isomorphism, let $(\alpha_1, \ldots, \alpha_d) \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}, X)$, $k \in \mathbb{Z}_d$, and denote the components of $\alpha_k = \begin{pmatrix} \alpha_{k1} & \alpha_{k2} & \ldots & \alpha_{kd} \end{pmatrix} \in \mathrm{Hom}_S(S^d, F_k)$ as in Lemma 3.1.10. Furthermore, Lemma 3.1.10 tells us that $\alpha_{k(k+j)} = \theta_{k(k+j)} \alpha_{(k+j)(k+j)}$ for each $j \neq 0$. Hence, $\alpha_k = \begin{pmatrix} \theta_{k1}\alpha_{11} & \theta_{k2}\alpha_{22} & \cdots & \theta_{kd}\alpha_{dd} \end{pmatrix}$. It follows that the morphism

$(\alpha_1, \ldots, \alpha_d)$ depends only on the diagonal components $\alpha_{11}, \alpha_{22}, \ldots, \alpha_{dd}$. In particular, the tuple $(\alpha_{11}, \ldots, \alpha_{dd}) \in \bigoplus_{j=1}^{d} F_j$ is a pre-image for $(\alpha_1, \ldots, \alpha_d)$ under the map θ. Finally, θ is injective since k-th component of $\begin{pmatrix} \theta_{k1}g_1 & \cdots & \theta_{kk}g_k & \cdots & \theta_{kd}g_d \end{pmatrix}$ is $\theta_{kk}g_k = g_k$. □

Corollary 3.1.12. *For any $X \in \mathrm{MF}_S^d(f)$ of size n, the Γ-module $\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}, X)$ is a free S-module of rank dn. In particular, $\mathcal{F}(X) \in \mathrm{MCM}(\Gamma)$.* □

Consider how the elements $e_{ii} \in \Gamma$ act on a morphism $\alpha : \mathcal{P} \to X$. From the Lemma 3.1.11, we may write $\alpha_k = \begin{pmatrix} \theta_{k1}g_1 & \cdots & \theta_{kd}g_d \end{pmatrix}$ for some $(g_1, \ldots, g_d) \in \bigoplus_{j=1}^{d} F_j$. For $i \in \mathbb{Z}_d$, we write $e_{ii} = (\epsilon_{ii}^1, \ldots, \epsilon_{ii}^d)$ where $\epsilon_{ii}^k(a_1, \ldots, a_d) = (0, \ldots, 0, a_i, 0, \ldots, 0)$ for any $(a_1, \ldots, a_d) \in S^d$. It follows that $\alpha_k \circ \epsilon_{ii}^k = \begin{pmatrix} 0 & \cdots & \theta_{ki}g_i & \cdots & 0 \end{pmatrix}$ where the only nonzero entry is in the i-th position. Hence, $e_{ii} \cdot \alpha = \left(\begin{pmatrix} 0 & \cdots & \theta_{ki}g_i & \cdots & 0 \end{pmatrix}\right)_{k=1}^{d} \in e_{ii}\mathcal{F}(X)$ and in fact this is the form of every element of $e_{ii}\mathcal{F}(X)$:

$$e_{ii}\mathcal{F}(X) = \left\{ \left(\begin{pmatrix} 0 & \cdots & \theta_{ki}g_i & \cdots & 0 \end{pmatrix}\right)_{k=1}^{d} \middle| g_i \in F_i \right\}.$$

Proposition 3.1.13. *Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$. Then $(\theta|_{F_1}, \ldots, \theta|_{F_d}) : X \to \mathcal{HF}(X)$ is an isomorphism of matrix factorizations.*

Proof. First notice that for any $g_i \in F_i$,

$$\theta|_{F_i}(g_i) = \theta(0, \ldots, 0, g_i, 0, \ldots, 0) = \left(\begin{pmatrix} 0 & \cdots & \theta_{ki}g_i & \cdots & 0 \end{pmatrix}\right)_{k=1}^{d} \in e_{ii}\mathcal{F}(X).$$

The restriction of θ is injective by definition and surjective by the paragraph preceding the proposition. Therefore, it is enough to show the commutativity of the diagram

$$\begin{array}{ccccccccc} F_1 & \xrightarrow{\varphi_d} & F_d & \xrightarrow{\varphi_{d-1}} & \cdots & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\varphi_1} & F_1 \\ \downarrow{\scriptstyle \theta|_{F_1}} & & \downarrow{\scriptstyle \theta|_{F_d}} & & & & \downarrow{\scriptstyle \theta|_{F_2}} & & \downarrow{\scriptstyle \theta|_{F_1}} \\ e_{11}\mathcal{F}(X) & \xrightarrow[z]{} & e_{dd}\mathcal{F}(X) & \xrightarrow[z]{} & \cdots & \xrightarrow[z]{} & e_{22}\mathcal{F}(X) & \xrightarrow[z]{} & e_{11}\mathcal{F}(X). \end{array}$$

Let $i \in \mathbb{Z}_d$ and $g_{i+1} \in F_{i+1}$. Then $\theta|_{F_i}\varphi_i(g_{i+1}) = \left(\begin{pmatrix} 0 & \cdots & \theta_{ki}\varphi_i(g_{i+1}) & \cdots & 0 \end{pmatrix}\right)_{k=1}^{d}$ and

$$\theta_{ki}\varphi_i(g_{i+1}) = \begin{cases} fg_{i+1} & k = i+1 \\ \varphi_k\varphi_{k+1}\cdots\varphi_{i-1}\varphi_i(g_{i+1}) & k \neq i+1. \end{cases}$$

To compute the other composition, recall that $z \cdot e_{(i+1)(i+1)}\alpha = e_{(i+1)i}\alpha$ for any $\alpha \in \mathcal{F}(X)$. Write $e_{(i+1)i} = (\epsilon^1_{(i+1)i}, \epsilon^2_{(i+1)i}, \ldots, \epsilon^d_{(i+1)i})$ where $\epsilon^k_{(i+1)i} : S^d \to S^d$ for each $k \in \mathbb{Z}_d$. It suffices to compute the composition of S-homomorphisms $S^d \to F_k$

$$\left(\begin{pmatrix} 0\cdots & \theta_{k(i+1)}(g_i)\cdots & 0 \end{pmatrix} \circ \epsilon^k_{(i+1)i}\right)(a_1, \ldots, a_d),$$

for each $k \in \mathbb{Z}_d$ and $(a_1, \ldots, a_d) \in S^d$. We have that

$$\epsilon^k_{(i+1)i}(a_1, \ldots, a_d) = \begin{cases} (0, \ldots, fa_i, \ldots, 0) & k = i+1 \\ (0, \ldots, a_i, \ldots, 0) & k \neq i+1 \end{cases}$$

where the only non-zero entries are in the $(i+1)$st position. Thus, the composition above is equal to $a_i\theta_{k(i+1)}(g_i)$ when $k \neq i+1$ and fa_ig_i when $k = i+1$. Comparing this with the components of $\theta|_{F_i}\varphi_i(g_{i+1})$ we conclude that $z \circ \theta|_{F_{i+1}}(g_{i+1}) = \theta|_{F_i} \circ \varphi_i(g_{i+1})$. Hence, $(\theta|_{F_1}, \ldots, \theta|_{F_d}) : X \to \mathcal{HF}(X)$ is an isomorphism of matrix factorizations. □

As a consequence of Proposition 3.1.13, the functor $\mathcal{H}$ is also dense. This completes the proof of Theorem 3.1.6. It is also worth mentioning that the analogous statement for the composition $\mathcal{FH}$ is true, that is, $\mathcal{FH}(M) \cong M$ for any $M \in \mathrm{MCM}(\Gamma)$. This follows from observing that the isomorphism of free S-modules, $\theta_{\mathcal{H}(M)} : M \to \mathcal{FH}(M)$, is also a Γ-homomorphism. As in the proof of Proposition 3.1.9, one can show that $e_{i(i-1)}\theta_{\mathcal{H}(M)}(m) = \theta_{\mathcal{H}(M)}(e_{i(i-1)}m)$ for all $m \in M$ and $i \in \mathbb{Z}_d$.

3.2 Periodic Resolutions

Let $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring, $f \in S$ a non-zero non-unit, and $d \geq 2$. An important consequence of Eisenbud's Theorem 1.2.7 is that the minimal free resolution of any finitely generated module over the hypersurface ring $R = S/(f)$ is eventually periodic with period at most two. The periodic part of the resolution is precisely given by a matrix factorization (with 2 factors). Moreover, this property characterizes hypersurface rings: If a local ring R has the property that the minimal free resolution of every finitely generated module is eventually periodic, then the completion of R must be a hypersurface ring (see [Eis80, Corollary 6.2]).

With this characterization in mind, we make two observations about the ring $\Gamma = \mathrm{End}_{\mathrm{MF}_S^d(f)}(\mathcal{P})^{\mathrm{op}}$ defined in Section 3.1.

Proposition 3.2.1. *Let $\mathcal{P} = \bigoplus_{i \in \mathbb{Z}_d} \mathcal{P}_i$ and $\Gamma = \mathrm{End}_{\mathrm{MF}_S^d(f)}(\mathcal{P})^{\mathrm{op}}$. Then every finitely generated left Γ-module has a projective resolution which is eventually periodic of period at most 2.*

Proof. Let N be a finitely generated Γ-module and set $r = \dim S$. Let $M = \mathrm{syz}_\Gamma^r(N)$ be an arbitrary r-th syzygy of N over Γ, and let

$$0 \longrightarrow M \longrightarrow P_{r-1} \longrightarrow P_{r-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

be the first $r-1$ steps of a projective resolution of N for some finitely generated projective Γ-modules P_i, $i = 0, 1, \ldots, r-1$. Recall that finitely generated projective Γ-modules are in $\mathrm{MCM}(\Gamma)$, that is, they are finitely generated free S-modules. Thus, the Depth Lemma implies that $\mathrm{depth}_S(M) = r$. Since MCM S-modules are free, we have that $M \in \mathrm{MCM}(\Gamma)$ as well. Now, by Section 3.1, there exists $X \in \mathrm{MF}_S^d(f)$ of size n such that $\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(\mathcal{P}, X) \cong M$. Since $\mathcal{P}$ is projective in $\mathrm{MF}_S^d(f)$, the functor $\mathcal{F}$ is exact. In particular, applying $\mathcal{F}$ to the periodic resolution constructed in Proposition 2.3.8 yields an exact sequence of MCM Γ-modules

$$\cdots \xrightarrow{\mathcal{F}(p)} \mathcal{F}(P(X)) \xrightarrow{\mathcal{F}(q)} \mathcal{F}(I(X)) \xrightarrow{\mathcal{F}(p)} \mathcal{F}(P(X)) \longrightarrow M \longrightarrow 0.$$

Actually, this is a free resolution of M over Γ since $\mathcal{F}(P(X)) \cong \mathcal{F}(\bigoplus_{i\in\mathbb{Z}_d} \mathcal{P}_i^n) \cong \Gamma^n$ and similarly $\mathcal{F}(I(X)) \cong \Gamma^n$. Thus, splicing together this periodic free resolution of M and the projective resolution of N, we get an eventually periodic resolution of N with period at most 2. ∎

Recall that a Noetherian ring Λ is said to be *Iwanaga-Gorenstein* if $\operatorname{injdim}_\Lambda \Lambda$ and $\operatorname{injdim}_{\Lambda^{\mathrm{op}}} \Lambda$ are both finite.

Lemma 3.2.2. *Let r be the Krull dimension of S. The ring $\Gamma = \operatorname{End}_{\mathrm{MF}_S^d(f)}(\mathcal{P})^{\mathrm{op}}$ is Iwanaga-Gorenstein of dimension r.*

Proof. First we show that any short exact sequence

$$0 \longrightarrow \Gamma \xrightarrow{q} M \xrightarrow{p} M' \longrightarrow 0 \tag{3.2.1}$$

with $M, M' \in \mathrm{MCM}(\Gamma)$ splits. To see this, first note that $\mathcal{H}(\Gamma) = \mathcal{H}\mathcal{F}(\mathcal{P}) \cong \mathcal{P}$ is injective in $\mathrm{MF}_S^d(f)$. Therefore, the short exact sequence of matrix factorizations

$$\mathcal{H}(\Gamma) \rightarrowtail^{\mathcal{H}(q)} \mathcal{H}(M) \xrightarrow{\mathcal{H}(p)} \mathcal{H}(M')$$

is split. Since $\mathcal{H}$ is full and faithful, there exists $t : M \to \Gamma$ such that $\mathcal{H}(t)\mathcal{H}(q) = 1_{\mathcal{H}(\Gamma)}$ and $tq = 1_\Gamma$ which implies that (3.2.1) is split.

To finish the proof, we apply results from [Aus86] which apply to both Γ and Γ^{op}. By [Aus86, Lemma 1.1] we have that $\Gamma \cong \operatorname{Hom}_S(Q, S)$ for some projective Γ^{op}-module Q. The functor $\operatorname{Hom}_S(\square, S) : \mathrm{MCM}(\Gamma) \to \mathrm{MCM}(\Gamma^{\mathrm{op}})$ defines a duality and therefore

$$Q \cong \operatorname{Hom}_S(\operatorname{Hom}_S(Q, S), S) \cong \operatorname{Hom}_S({}_\Gamma\Gamma, S).$$

Thus, $\mathrm{Hom}_S({}_\Gamma\Gamma, S)$ is Γ^{op}-projective which happens if and only if $\mathrm{Hom}_S({}_{\Gamma^{\mathrm{op}}}\Gamma, S)$ is Γ-projective according to [Aus86, Lemma 5.1]. In this case, [Aus86, p. 5.2] says that $\mathrm{injdim}_\Gamma \Gamma = \mathrm{injdim}_S S$ which is equal to r since S is Gorenstein. Interchanging the roles of Γ and Γ^{op} we find that $\mathrm{injdim}_{\Gamma^{\mathrm{op}}} \Gamma = r$ as well. $\square$

Corollary 3.2.1 and Lemma 3.2.2 give a homological description of Γ which resembles that of a commutative hypersurface ring. This prompts us to call Γ a "non-commutative hypersurface ring".

4 | Branched Covers

4.1 The d-fold branched cover

Let $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring, and let f be a non-zero non-unit in S. Set $R = S/(f)$ and fix an integer $d \geq 2$.

Definition 4.1.1. The *(d-fold) branched cover* of R is the hypersurface ring

$$R^\sharp = S[\![z]\!]/(f + z^d).$$

Throughout this chapter, we will also assume that $\mathbf{k}$ is algebraically closed and that the characteristic of $\mathbf{k}$ does not divide d. In this case, the polynomial $x^d - 1 \in \mathbf{k}[x]$ has d distinct roots in $\mathbf{k}$ and the group formed by its roots is cyclic of order d. Any generator of this group is a *primitive d-th root of unity*. Since S is complete, it also contains primitive d-th roots of $1 \in S$ [LW12, Corollary A.31].

Fix an element $\omega \in S$ such that $\omega^d = 1$ and $\omega^t \neq 1$ for all $0 < t < d$. The ring $R^\sharp$ carries an automorphism $\sigma : R^\sharp \to R^\sharp$ of order d which fixes S and sends z to ωz. Denote by $R^\sharp[\sigma]$ the *skew group algebra* of the cyclic group of order d generated by σ acting on $R^\sharp$. That is, $R^\sharp[\sigma] = \bigoplus_{i \in \mathbb{Z}_d} R^\sharp \cdot \sigma^i$ as $R^\sharp$-modules with multiplication given by the rule

$$(s \cdot \sigma^i) \cdot (t \cdot \sigma^j) = s\sigma^i(t) \cdot \sigma^{i+j}$$

for $s, t \in R^\sharp$ and $i, j \in \mathbb{Z}_d$. The left modules over $R^\sharp[\sigma]$ are precisely the $R^\sharp$-modules N which carry a compatible action of σ, that is, an action of σ such that $\sigma(rx) = \sigma(r)\sigma(x)$ for all $r \in R^\sharp$ and $x \in N$. It follows that $R^\sharp$ itself is naturally a left $R^\sharp[\sigma]$-module with the action of σ given by evaluating $\sigma(r)$ for any $r \in R^\sharp$. We say that a left $R^\sharp[\sigma]$-module is *maximal Cohen-Macaulay* (MCM as usual) if it is MCM when it is viewed as an $R^\sharp$-module. Denote the category of MCM $R^\sharp[\sigma]$-modules by $\mathrm{MCM}_\sigma(R^\sharp)$.

In the case $d = 2$, Knörrer showed that the category of MCM modules over $R^\sharp[\sigma]$ is equivalent to the category of matrix factorizations of f with 2 factors [Knö87, Proposition 2.1]. The main goal of this section is to extend the equivalence given by Knörrer for all $d \geq 2$ (Theorem 4.1.5).

Lemma 4.1.2. *Let N be an $R^\sharp[\sigma]$-module. Then N decomposes as an S-module into $N = \bigoplus_{i \in \mathbb{Z}_d} N^{\omega^i}$ where*

$$N^{\omega^i} = \left\{ x \in N : \sigma(x) = \omega^i x \right\}.$$

Furthermore, if N is a MCM $R^\sharp[\sigma]$-module, then N and each summand N^{ω^i} are finitely generated free S-modules.

Proof. In order to justify the direct sum decomposition of N, we will make repeated use of the fact $\sum_{i=0}^{d-1} \omega^{ki} = 0$ for any $k \in \mathbb{Z}_d$. Let $x \in N$ and observe that

$$\begin{aligned} dx &= dx + \left(\sum_{i=0}^{d-1} \omega^{-i} \right) \sigma(x) + \left(\sum_{i=0}^{d-1} \omega^{-2i} \right) \sigma^2(x) + \cdots + \left(\sum_{i=0}^{d-1} \omega^{-(d-1)i} \right) \sigma^{d-1}(x) \\ &= \sum_{i=0}^{d-1} \sigma^i(x) + \sum_{i=0}^{d-1} \omega^{-i} \sigma^i(x) + \cdots + \sum_{i=0}^{d-1} \omega^{-(d-1)i} \sigma^i(x) \\ &= \sum_{k=0}^{d-1} \sum_{i=0}^{d-1} \omega^{-ik} \sigma^i(x). \end{aligned}$$

Also notice that, for any $k \in \mathbb{Z}_d$,

$$\begin{aligned} \sigma \left(\sum_{i=0}^{d-1} \omega^{-ik} \sigma^i(x) \right) &= \sigma(x) + \omega^{-k} \sigma^2(x) + \omega^{-2k} \sigma^3(x) + \cdots + \omega^{-(d-1)k} x \\ &= \omega^k (x + \omega^{-k} \sigma(x) + \omega^{-2k} \sigma^2(x) + \cdots + \omega^{-(d-1)k} \sigma^{d-1}(x)). \end{aligned}$$

That is, $\sum_{i=0}^{d-1} \omega^{-ik}\sigma^i(x) \in N^{\omega^k}$. Since σ is S-linear and d is invertible in S, we have that

$$x = \sum_{k=0}^{d-1}\sum_{i=0}^{d-1} \frac{\omega^{-ik}\sigma^i(x)}{d} \in N^1 + N^{\omega} + \cdots + N^{\omega^{d-1}}$$

implying that $N = \sum_{k=0}^{d-1} N^{\omega^k}$.

Next, suppose we have a sum of elements

$$x_0 + x_1 + \cdots + x_{d-1} = 0 \tag{4.1.1}$$

with $x_i \in N^{\omega^i}$ for each $i \in \mathbb{Z}_d$, and let $j \in \mathbb{Z}_d$. Notice that if $k, \ell \in \mathbb{Z}_d$, then $\omega^{-jk}\sigma^k(x_\ell) = \omega^{-jk+k\ell}x_\ell = \omega^{(-j+\ell)k}x_\ell$. In particular, $\omega^{-jk}\sigma^k(x_j) = x_j$ for all $k \in \mathbb{Z}_d$. Therefore, applying $\omega^{-jk}\sigma^k$ to (4.1.1) gives us an equation

$$\omega^{-jk}x_0 + \omega^{(-j+1)k}x_1 + \cdots + x_j + \cdots + \omega^{(-j-1)k}x_{d-1} = 0.$$

Summing over $\mathbb{Z}_d$, we find that

$$\sum_{i \neq j}\sum_{k \in \mathbb{Z}_d} \omega^{k(-j+i)}x_i + dx_j = 0.$$

Once again, since $\sum_{k=0}^{d-1} \omega^{k(-j+i)} = 0$ for all $i \neq j$, we can conclude that $x_j = 0$. Thus, $N = \bigoplus_{i=0}^{d-1} N^{\omega^i}$ as desired.

The second statement holds since a finitely generated $R^\sharp$-module N is MCM over $R^\sharp$ if and only if it is free as an S-module. □

As an S-module, $R^\sharp$ is finitely generated and free with basis given by $\{1, z, z^2, \ldots, z^{d-1}\}$. Consequently, a finitely generated $R^\sharp$-module N is MCM over $R^\sharp$ if and only if it is free over S [Yos90, Proposition 1.9]. Furthermore, multiplication by z on N defines an S-linear map $\varphi : N \to N$ which satisfies $\varphi^d = -f \cdot 1_N$. Conversely, given a free S-module F and a homomorphism $\varphi : F \to F$ satisfying $\varphi^d = -f \cdot 1_F$, the pair (F, φ) defines an MCM $R^\sharp$-

module whose z-action is given by the map φ. We will use these perspectives interchangeably throughout the rest of this chapter.

Definition and Proposition 4.1.3. Let $R, R^\sharp$, $R^\sharp[\sigma]$, and ω be as above. Let $\mu \in S$ be any root of $x^d + 1 \in S[x]$.

(i) Let N be an MCM $R^\sharp[\sigma]$-module and N^{ω^i} be as in Lemma 4.1.2 for each $i \in \mathbb{Z}_d$. Define a matrix factorization $\mathcal{A}(N) \in \mathrm{MF}^d_S(f)$ as follows. Multiplication by μz defines an S-linear homomorphism

$$N^{\omega^i} \to N^{\omega^{i+1}}$$

for all $i \in \mathbb{Z}_d$. The composition

$$N^{\omega^{d-1}} \xrightarrow{\mu z} N^1 \xrightarrow{\mu z} N^\omega \xrightarrow{\mu z} \cdots \xrightarrow{\mu z} N^{\omega^{d-2}} \xrightarrow{\mu z} N^{\omega^{d-1}}$$

is equal to $-z^d = f$ times the identity on $N^{\omega^{d-1}}$. It follows that the above homomorphisms and free S-modules form a matrix factorization of f in $\mathrm{MF}^d_S(f)$ which we denote as $\mathcal{A}(N)$. For a homomorphism $g : N \to M$ of MCM $R^\sharp[\sigma]$-modules, define a morphism of matrix factorizations

$$\mathcal{A}(g) = \left(g|_{N^{\omega^{d-1}}}, g|_{N^{\omega^{d-2}}}, \ldots, g|_{N^1}\right)$$

where $g|_{N^{\omega^i}}$ denotes the restriction of g to the S-direct summand N^{ω^i} of N. Thus, we have a functor $\mathcal{A} : \mathrm{MCM}_\sigma(R^\sharp) \to \mathrm{MF}^d_S(f)$.

(ii) Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}^d_S(f)$. Define

$$\mathcal{B}(X) = F_d \oplus F_{d-1} \oplus \cdots \oplus F_1$$

as an S-module. Give $\mathcal{B}(X)$ the structure of a $R^\sharp[\sigma]$-module by defining the action of

z as

$$z \cdot (x_d, x_{d-1}, \ldots, x_1) = \big(\mu^{-1}\varphi_d(x_1), \mu^{-1}\varphi_{d-1}(x_d), \ldots, \mu^{-1}\varphi_1(x_2)\big)$$

and the action of σ as

$$\sigma \cdot (x_d, x_{d-1}, \ldots, x_1) = (x_d, \omega x_{d-1}, \omega^2 x_{d-2}, \ldots, \omega^{d-1} x_1),$$

for any $x_i \in F_i$, $i \in \mathbb{Z}_d$. For a morphism of matrix factorizations $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) : X \to X'$, where $X' = (\varphi_1' : F_2' \to F_1', \ldots, \varphi_d' : F_1' \to F_d')$, define $\mathcal{B}(\alpha) : \mathcal{B}(X) \to \mathcal{B}(X')$ by

$$\mathcal{B}(\alpha)(x_d, x_{d-1}, \ldots, x_1) = (\alpha_d(x_d), \alpha_{d-1}(x_{d-1}), \ldots, \alpha_1(x_1))$$

for all $(x_d, x_{d-1} \ldots, x_1) \in \mathcal{B}(X)$. Thus, we have a functor $\mathcal{B} : \mathrm{MF}_S^d(f) \to \mathrm{MCM}_\sigma(R^\sharp)$.

Proof. Several pieces of the definitions need justification. First we note that, since -1 has a d-th root in $\mathbf{k}$, we may apply [LW12, Corollary A.31] to obtain an element $\mu \in S$ such that $\mu^d = -1$.

(i) Multiplication by μz defines an S-linear map $N^{\omega^i} \to N^{\omega^{i+1}}$ for any $i \in \mathbb{Z}_d$ since $\mu \in S$ and

$$\sigma(zx) = \sigma(z)\sigma(x) = \omega^{i+1} zx$$

for all $x \in N^{\omega^i}$. Notice that $(\mu z)^d = \mu^d z^d = f \in R^\sharp$. Therefore, the composition

$$N^{\omega^{i+1}} \xrightarrow{(\mu z)^{d-1}} N^{\omega^i} \xrightarrow{\mu z} N^{\omega^{i+1}}$$

equals $f \cdot 1_{N^{\omega^i}}$ for all $i \in \mathbb{Z}_d$. Similarly, the composition

$$N^{\omega^i} \xrightarrow{\mu z} N^{\omega^{i+1}} \xrightarrow{(\mu z)^{d-1}} N^{\omega^i}$$

equals $f \cdot 1_{N^{\omega^{i+1}}}$ for all $i \in \mathbb{Z}_d$. We know that each N^{ω^i} is a free S-module by Lemma

4.1.2 so, by applying Lemma 1.2.9, we have that $\operatorname{rank}_S(N^{\omega^i}) = \operatorname{rank}_S(N^{\omega^{i+1}})$ for all $i \in \mathbb{Z}_d$. This implies that $\mathcal{A}(N) \in \mathrm{MF}_S^d(f)$.

If $g : N \to M$ is a homomorphism of $R^\sharp[\sigma]$-modules, then $g(rx) = rg(x)$ and $\sigma g(x) = g(\sigma(x))$ for all $x \in N$ and $r \in R^\sharp$. It follows that $g|_{N^{\omega^i}}(N^{\omega^i}) \subseteq M^{\omega^i}$ and that the diagram

$$\begin{array}{ccc} N^{\omega^i} & \xrightarrow{\mu z} & N^{\omega^{i+1}} \\ {\scriptstyle g|_{N^{\omega^i}}}\downarrow & & \downarrow{\scriptstyle g|_{N^{\omega^{i+1}}}} \\ M^{\omega^i} & \xrightarrow{\mu z} & M^{\omega^{i+1}} \end{array}$$

commutes for all $i \in \mathbb{Z}_d$. In other words, $\mathcal{A}(g)$ is a morphism of matrix factorizations $\mathcal{A}(N) \to \mathcal{A}(M)$.

(ii) First we justify that the defined actions of z and σ make $\mathcal{B}(X)$ a MCM $R^\sharp[\sigma]$-module. Recall the homomorphisms $\theta_{ki}^X : F_i \to F_k$, from Section 2.1, which are given by

$$\theta_{ki}^X = \begin{cases} 1_{F_k} & i = k \\ \varphi_k \varphi_{k+1} \cdots \varphi_{i-2} \varphi_{i-1} & i \neq k. \end{cases}$$

We will drop the superscript X for the rest of this proof. We claim that for any $s \geq 1$ and $(x_d, x_{d-1}, \ldots, x_1) \in \mathcal{B}(X)$,

$$z^s \cdot (x_d, x_{d-1}, \ldots, x_1) = f^q \mu^{-r} (\theta_{d(d+r)}(x_{d+r}), \ldots, \theta_{1(1+r)}(x_{1+r}))$$

where $s = dq + r$, $q \geq 0$, and $0 \leq r < d$. When $s = 1$ the formula is precisely the defined action of z on $\mathcal{B}(X)$. Assume the claim is true for $s = dq + r \geq 1$ with $q \geq 0$ and $0 \leq r < d$ and consider multiplication by z^{s+1}. By induction we have that

$$\begin{aligned} z^{s+1} \cdot (x_d, \ldots, x_1) &= z \cdot f^q \mu^{-r} ((\theta_{d(d+r)}(x_{d+r}), \ldots, \theta_{1(1+r)}(x_{1+r})) \\ &= f^q \mu^{-(r+1)} (\varphi_d \theta_{1(1+r)}(x_{1+r}), \ldots, \varphi_1 \theta_{2(2+r)}(x_{2+r})). \end{aligned}$$

If $r = d-1$, then $\varphi_{k-1}\theta_{k(k+r)} = f \cdot 1_{F_{k-1}}$ for each $k \in \mathbb{Z}_d$ and therefore

$$z^{s+1} = f^{q+1}\mu^{-(r+1)}(x_d, x_{d-1}, \ldots, x_1).$$

If $0 \le r < d-1$, then $0 \le r+1 < d$ and therefore $\varphi_{k-1}\theta_{k(k+r)} = \theta_{(k-1)(k+r)}$ for each $k \in \mathbb{Z}_d$. In this case,

$$z^{s+1} = f^q \mu^{-(r+1)}(\theta_{d(1+r)}(x_{1+r}), \ldots, \theta_{1(2+r)}(x_{2+r}))$$

which completes the induction. It follows that multiplication by z^d is given by

$$z^d \cdot (x_d, \ldots, x_1) = f\mu^{-d}(x_d, \ldots, x_1).$$

By definition, $\mu^{-d} = -1$. Thus, $(f + z^d)\mathcal{B}(X) = 0$, that is, $\mathcal{B}(X)$ is an $R^\sharp$-module. In fact, since $\mathcal{B}(X)$ is free as an S-module, it is MCM as an $R^\sharp$-module.

In order to show that $\mathcal{B}(X)$ has the structure of an $R^\sharp[\sigma]$-module, we must show that $\sigma(rx) = \sigma(r)\sigma(x)$ for all $r \in R^\sharp$ and $x \in \mathcal{B}(X)$. It suffices to show that $\sigma(zx) = \sigma(z)\sigma(x)$ for all $x \in \mathcal{B}(X)$. This follows since

$$\begin{aligned}
\sigma(z)\sigma(x) &= \omega z \cdot (x_d, \omega x_{d-1}, \ldots, \omega^{d-1}x_1) \\
&= z \cdot (\omega x_d, \omega^2 x_{d-1}, \ldots, x_1) \\
&= (\mu^{-1}\varphi_d(x_1), \mu^{-1}\omega\varphi_{d-1}(x_d), \ldots, \mu^{-1}\omega^{d-1}\varphi_1(x_2)) \\
&= \sigma\left(\mu^{-1}\varphi_d(x_1), \mu^{-1}\varphi_{d-1}(x_d), \ldots, \mu^{-1}\varphi_1(x_2)\right) \\
&= \sigma(zx)
\end{aligned}$$

for any $x = (x_d, x_{d-1}, \ldots, x_1) \in \mathcal{B}(X)$. Hence, $\mathcal{B}(X) \in \mathrm{MCM}_\sigma(R^\sharp)$.

Finally, we must show that $\mathcal{B}(\alpha)$ forms a homomorphism of $R^\sharp[\sigma]$-modules. This is straightforward to verify by recalling that $\alpha_k\varphi_k = \varphi'_k\alpha_{k+1}$ for all $k \in \mathbb{Z}_d$. □

Remark 4.1.4. The role of μ in the definition of $\mathcal{A}(N)$ is to obtain a d-fold factorization of f (instead of $-f$) and to do so in a symmetric way. It is important to note that the isomorphism class of $\mathcal{A}(N) \in \mathrm{MF}_S^d(f)$ is independent of the choice of μ. To see this, observe that given another root μ' of x^d+1, we may write $\mu' = \omega^j \mu$ for some $j \in \mathbb{Z}_d$ and obtain an isomorphism of matrix factorizations:

$$\begin{array}{ccccccccccc}
N^{\omega^{d-1}} & \xrightarrow{\mu z} & N^1 & \xrightarrow{\mu z} & N^{\omega} & \xrightarrow{\mu z} & \cdots & \xrightarrow{\mu z} & N^{\omega^{d-2}} & \xrightarrow{\mu z} & N^{\omega^{d-1}} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \omega^{-(d-1)j}} & & \downarrow{\scriptstyle \omega^{-(d-2)j}} & & & & \downarrow{\scriptstyle \omega^{-j}} & & \downarrow{\scriptstyle 1} \\
N^{\omega^{d-1}} & \xrightarrow{\mu' z} & N^1 & \xrightarrow{\mu' z} & N^{\omega} & \xrightarrow{\mu' z} & \cdots & \xrightarrow{\mu' z} & N^{\omega^{d-2}} & \xrightarrow{\mu' z} & N^{\omega^{d-1}}.
\end{array}$$

Similarly, μ^{-1} in the definition of $X^\sharp$ ensures that we obtain a module over $R^\sharp$ and the isomorphism class of $X^\sharp$ in $\mathrm{MCM}(R^\sharp)$ is also independent of the choice of μ.

Theorem 4.1.5. *The functors* $\mathcal{A} : \mathrm{MCM}_\sigma(R^\sharp) \to \mathrm{MF}_S^d(f)$ *and* $\mathcal{B} : \mathrm{MF}_S^d(f) \to \mathrm{MCM}_\sigma(R^\sharp)$ *are naturally inverse and establish an equivalence of the categories* $\mathrm{MCM}_\sigma(R^\sharp) \approx \mathrm{MF}_S^d(f)$.

Proof. Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$. Then $\mathcal{B}(X) = F_d \oplus F_{d-1} \oplus \cdots \oplus F_1$ with the action of σ on $\mathcal{B}(X)$ given by $\sigma(x_d, \ldots, x_1) = (x_d, \omega x_{d-1}, \ldots, \omega^{d-1} x_1)$ for each $x_i \in F_i$. For each $i \in \mathbb{Z}_d$, the S-module F_i is embedded into $\mathcal{B}(X)$ via the natural inclusion map which we will denote as $q_i : F_i \to \mathcal{B}(X)$. Notice that the action of σ on $\mathcal{B}(X)$ implies that

$$\mathcal{B}(X)^{\omega^{d-i}} = \{(0, \ldots, 0, x_i, 0, \ldots, 0) : x_i \in F_i\} = q_i(F_i).$$

Therefore, the matrix factorization $\mathcal{AB}(X)$ is given by

$$\mathcal{B}(X)^{\omega^{d-1}} \xrightarrow{\mu z} \mathcal{B}(X)^1 \xrightarrow{\mu z} \cdots \xrightarrow{\mu z} \mathcal{B}(X)^{\omega^{d-2}} \xrightarrow{\mu z} \mathcal{B}(X)^{\omega^{d-1}}$$

which is isomorphic to X via the isomorphism

$$\begin{array}{ccccccccc}
F_1 & \xrightarrow{\varphi_d} & F_d & \xrightarrow{\varphi_{d-1}} & \cdots & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\varphi_1} & F_1 \\
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_d} & & & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_1} \\
\mathcal{B}(X)^{\omega^{d-1}} & \xrightarrow{\mu z} & \mathcal{B}(X)^1 & \xrightarrow{\mu z} & \cdots & \xrightarrow{\mu z} & \mathcal{B}(X)^{\omega^{d-2}} & \xrightarrow{\mu z} & \mathcal{B}(X)^{\omega^{d-1}}.
\end{array}$$

Indeed, the diagram above commutes since

$$\mu z q_{k+1}(x) = \mu q_k(\mu^{-1}\varphi_k(x)) = q_k \varphi_k(x)$$

for all $k \in \mathbb{Z}_d$ and $x \in F_{k+1}$.

To show $\mathcal{AB}$ is naturally isomorphic to the identity, suppose we have a morphism $\alpha = (\alpha_1, \ldots, \alpha_d) : X \to X'$ where $X' = (\varphi'_1 : F'_2 \to F'_1, \ldots, \varphi'_d : F'_1 \to F'_d) \in \mathrm{MF}^d_S(f)$. The matrix factorizations X' is isomorphic to $\mathcal{AB}(X')$ via the morphism $(q'_1, q'_2, \ldots, q'_d)$ where $q'_i : F'_i \to \mathcal{B}(X')$ is the natural inclusion. Recall that the homomorphism $\mathcal{B}(\alpha)$ is given by

$$\mathcal{B}(\alpha)(x_d, x_{d-1}, \ldots, x_1) = (\alpha_d(x_d), \alpha_{d-1}(x_{d-1}), \ldots, \alpha_1(x_1)).$$

Applying the functor $\mathcal{A}$ forms a morphism of matrix factorizations by restricting $\mathcal{B}(\alpha)$ to the submodules $\mathcal{B}(X)^{\omega^{d-i}}$. The images of these restrictions land in the submodules $\mathcal{B}(X')^{\omega^{d-i}}$. In other words, the k-th component of the morphism $\mathcal{AB}(\alpha)$ is given by the composition

$$\mathcal{B}(X)^{\omega^{d-k}} \xrightarrow{p_k} F_k \xrightarrow{\alpha_k} F'_k \xrightarrow{q'_k} \mathcal{B}(X')^{\omega^{d-k}}$$

where p_k is the natural projection onto F_k. Therefore,

$$\begin{aligned} \mathcal{AB}(\alpha) \circ (q_1, q_2, \ldots, q_d) &= (q'_1\alpha_1 p_1, q'_2\alpha_2 p_2, \ldots, q'_d\alpha_d p_d) \circ (q_1, q_2, \ldots, q_d) \\ &= (q'_1\alpha_1, q'_2\alpha_2, \ldots, q'_d\alpha_d) \end{aligned}$$

and this implies the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\alpha\quad} & X' \\ {\scriptstyle (q_1,\ldots,q_d)}\downarrow & & \downarrow{\scriptstyle (q'_1,\ldots,q'_d)} \\ \mathcal{AB}(X) & \xrightarrow{\mathcal{AB}(\alpha)} & \mathcal{AB}(X'). \end{array}$$

Next, let N be an MCM $R^\sharp[\sigma]$-module. As an S-module,

$$\mathcal{BA}(N) = N^1 \oplus N^\omega \oplus \cdots \oplus N^{\omega^{d-1}}.$$

In fact, the natural S-isomorphism $\Psi_N : \mathcal{BA}(N) \to N$ given by $(n_0, n_1, \ldots, n_{d-1}) \mapsto \sum_{i\in\mathbb{Z}_d} n_i$ is also an $R^\sharp[\sigma]$-homomorphism. To see this, let $(n_0, n_1, \ldots, n_{d-1}) \in \mathcal{BA}(N)$. Then Ψ_N is a $R^\sharp$-homomorphism since

$$\begin{aligned}\Psi_N(z\cdot(n_0, n_1, \ldots, n_{d-1})) &= \Psi_N(\mu^{-1}\mu z n_{d-1}, \mu^{-1}\mu z n_0, \ldots, \mu^{-1}\mu z n_{d-1})\\ &= \Psi_N(zn_{d-1}, zn_0, \ldots, zn_{d-1})\\ &= z(n_0 + n_1 + \cdots n_{d-1})\\ &= z\Psi_N(n_0, n_1, \ldots, n_{d-1})\end{aligned}$$

and a $R^\sharp[\sigma]$-homomorphism since

$$\begin{aligned}\Psi_N(\sigma(n_0, n_1, \ldots, n_{d-1})) &= \Psi_N(n_0, \omega n_1, \ldots, \omega^{d-1} n_{d-1})\\ &= n_0 + \omega n_1 + \cdots + \omega^{d-1} n_{d-1}\\ &= \sigma(n_0) + \sigma(n_1) + \cdots + \sigma(n_{d-1})\\ &= \sigma(n_0 + n_1 + \cdots n_{d-1})\\ &= \sigma(\Psi_N(n_0, n_1, \ldots, n_{d-1})).\end{aligned}$$

□

4.2 A ring isomorphism $R^\sharp[\sigma] \cong \Gamma$

In Section 3.1 we showed that the category of MCM modules over the endomorphism ring of the projective object $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_d$ is equivalent to the category of matrix factorizations of f with $d \geq 2$ factors. Together with Theorem 4.1.5 we have an induced equivalence of the module categories $\mathrm{MCM}_\sigma(R^\sharp)$ and $\mathrm{MCM}(\mathrm{End}_{\mathrm{MF}^d_S(f)}(\mathcal{P})^{\mathrm{op}})$. In fact, the two rings are isomorphic which we will see below. Note that this isomorphism only makes sense in the setting of this chapter since $R^\sharp[\sigma]$ is not well-defined otherwise.

Recall, also from Section 3.1, that $\Gamma = \mathrm{End}_{\mathrm{MF}^d_S(f)}(\mathcal{P})^{\mathrm{op}}$ is a free S-module with basis given by the elements $\{e_{ij}\}_{i,j\in\mathbb{Z}_d}$. The main rules for multiplication in Γ are given in Lemmas 3.1.3, 3.1.5, and 3.1.4.

Proposition 4.2.1. *The rings $R^\sharp[\sigma]$ and $\Gamma = \mathrm{End}_{\mathrm{MF}^d_S(f)}(\mathcal{P})^{\mathrm{op}}$ are isomorphic.*

Proof. The set $\{z^i\sigma^j\}_{i,j\in\mathbb{Z}_d}$ forms a basis for $R^\sharp[\sigma]$ over S. As in 4.1.3, let $\mu \in S$ be any root of $x^d + 1 \in S[x]$. Define a map $\psi : R^\sharp[\sigma] \to \Gamma$ by

$$\psi(z) = \mu \sum_{i\in\mathbb{Z}_d} e_{i(i-1)} \text{ and } \psi(\sigma) = \sum_{i\in\mathbb{Z}_d} \omega^{-i} e_{ii}.$$

Extend ψ multiplicatively, that is, define $\psi(z^i\sigma^j) = \psi(z)^i\psi(\sigma)^j$ for all $i, j \in \mathbb{Z}_d$. Since $\{z^i\sigma^j\}_{i,j\in\mathbb{Z}_d}$ is an S-basis, ψ extends uniquely to a well defined S-linear homomorphism.

We claim that ψ is also a ring homomorphism. Since ψ is S-linear, it suffices to check that $\psi(z^i\sigma^j \cdot z^\ell\sigma^k) = \psi(z^i\sigma^j) \cdot \psi(z^\ell\sigma^k)$ for all $i, j, \ell, k \in \mathbb{Z}_d$. From Lemma 3.1.3 we have that $e_{ii}e_{j(j-1)} = e_{j(j-1)}$ if $i = j - 1$ and 0 otherwise, and similarly, $e_{i(i-1)}e_{jj} = e_{i(i-1)}$ if $j = i$ and 0 otherwise. Therefore,

$$\begin{aligned}
\psi(\sigma)\psi(z) &= \Big(\sum_{i\in\mathbb{Z}_d} \omega^{-i} e_{ii}\Big)\Big(\mu \sum_{j\in\mathbb{Z}_d} e_{j(j-1)}\Big) \\
&= \mu \sum_{i\in\mathbb{Z}_d} \omega^{-(i-1)} e_{i(i-1)} \\
&= \mu\omega \sum_{i\in\mathbb{Z}_d} \omega^{-i} e_{i(i-1)} \\
&= \mu\omega\Big(\sum_{i\in\mathbb{Z}_d)} e_{i(i-1)}\Big)\Big(\sum_{j\in\mathbb{Z}_d} \omega^{-j} e_{jj}\Big) \\
&= \omega\psi(z)\psi(\sigma) \\
&= \psi(\omega z\sigma).
\end{aligned}$$

Since $\sigma z = \sigma(z)\sigma = \omega z\sigma$ in $R^\sharp[\sigma]$, we have that $\psi(\sigma z) = \psi(\sigma)\psi(z)$. By induction, it follows that $\sigma^i z^j = \omega^{ij} z^j \sigma^i$ and

$$\psi(\sigma^i z^j) = \omega^{ij}\psi(z)^j\psi(\sigma)^i = \psi(\sigma)^i\psi(z)^j$$

for all $i, j \in \mathbb{Z}_d$. The fact that ψ is a ring homomorphism now follows since

$$\begin{aligned}
\psi(z^i\sigma^j \cdot z^\ell\sigma^k) &= \psi(\omega^{\ell j} z^{i+\ell}\sigma^{j+k}) \\
&= \omega^{\ell j}\psi(z)^{i+\ell}\psi(\sigma)^{j+k} \\
&= \omega^{\ell j}\psi(z)^i\psi(z)^\ell\psi(\sigma)^j\psi(\sigma)^k \\
&= \psi(z)^i\psi(\sigma)^j\psi(z)^\ell\psi(\sigma)^k \\
&= \psi(z^i\sigma^j)\cdot\psi(z^\ell\sigma^k)
\end{aligned}$$

for all $i, j, \ell, k \in \mathbb{Z}_d$.

As S-modules, both $R^\sharp[\sigma]$ and Γ are free of rank d^2. Therefore, to conclude that ψ is an isomorphism, it suffices to check surjectivity. First, we show that the element e_{kk} is in the image of ψ for each $k \in \mathbb{Z}_d$. Indeed, if $j \in \mathbb{Z}_d$, then

$$\psi(\sigma^j) = \sum_{i\in\mathbb{Z}_d} \omega^{-ji} e_{ii}.$$

Thus, for any $k \in \mathbb{Z}_d$,

$$\begin{aligned}
\psi\left(\frac{1}{d}\sum_{j\in\mathbb{Z}_d}\omega^{jk}\sigma^j\right) &= \frac{1}{d}\sum_{j\in\mathbb{Z}_d}\omega^{jk}\psi(\sigma)^j \\
&= \frac{1}{d}\sum_{j\in\mathbb{Z}_d}\sum_{i\in\mathbb{Z}_d}\omega^{j(k-i)}e_{ii} \\
&= \frac{1}{d}\sum_{i\neq k}\sum_{j\in\mathbb{Z}_d}\omega^{j(k-i)}e_{ii} + \frac{1}{d}\sum_{j\in\mathbb{Z}_d}e_{kk} \\
&= e_{kk}.
\end{aligned}$$

Hence, the elements $e_{11}, e_{22}, \ldots, e_{dd}$, and $\sum_{i\in\mathbb{Z}_d} e_{i(i-1)}$ are in the image of ψ. It follows that $e_{k(k-1)} \in \operatorname{Im}\psi$ for all $k \in \mathbb{Z}_d$ since $e_{kk}\sum_{i\in\mathbb{Z}_d} e_{i(i-1)} = e_{k(k-1)}$ by Lemma 3.1.3 (iv). Finally, Lemma 3.1.4 allows us to conclude that $e_{ij} \in \operatorname{Im}\psi$ for all $i, j \in \mathbb{Z}_d$, implying that ψ is surjective as desired. □

4.3 Finite matrix factorization type

A local ring A is said to have *finite Cohen-Macaulay* (CM) *type* if, up to isomorphism, there are only finitely many indecomposable objects in the category $\mathrm{MCM}(A)$ of MCM A-modules. We adopt the following analogous terminology for the representation type of the category $\mathrm{MF}_S^d(f)$.

Definition 4.3.1. We say that f has *finite d-MF type* if the category $\mathrm{MF}_S^d(f)$ has, up to isomorphism, only finitely many indecomposable objects.

In [Knö87], Knörrer proved that $R = S/(f)$ has finite CM type if and only if $R^\sharp = S[\![z]\!]/(f+z^2)$ has finite CM type. The correspondence, given by Eisenbud [Eis80, Corollary 6.3], between matrix factorizations and MCM R-modules implies that the number of isomorphism classes of indecomposable objects in $\mathrm{MCM}(R)$ and $\mathrm{MF}_S^2(f)$ differ by only one. Since $R^\sharp$ is also a hypersurface ring, the same is true for $\mathrm{MCM}(R^\sharp)$ and $\mathrm{MF}^2_{S[\![z]\!]}(f+z^2)$. With this in mind, we state a version of Knörrer's theorem.

Theorem 4.3.2 ([Knö87], Corollary 2.8)**.** *Let $f \in \mathfrak{n}^2$ be non-zero, $R = S/(f)$, and $d = 2$ so that $R^\sharp = S[\![z]\!]/(f+z^2)$ and $\operatorname{char}\mathbf{k} \neq 2$. Then the following are equivalent:*

(i) R has finite CM type;

(ii) f has finite 2-MF type;

(iii) $R^\sharp$ has finite CM type;

(iv) $f+z^2$ has finite 2-MF type.

The main goal of this section is investigate which of the analogous implications for d-fold factorizations hold when $d \geq 2$. The rest of the results in the Chapter are joint with G. Leuschke and can be found in [LT21].

Our first observation is that the implications (ii) $\Longrightarrow$ (i) and (iv) $\Longrightarrow$ (iii) still hold for $d \geq 2$ in the following sense:

Lemma 4.3.3. *Let S be a regular local ring, f a non-zero non-unit in S, and $d \geq 2$. If f has finite d-MF type, then f has finite k-MF type for all $2 \leq k \leq d$. In particular, if f has finite d-MF type for some $d \geq 2$, then $R = S/(f)$ has finite CM type.*

Proof. It suffices to show that finite d-MF type implies finite $(d-1)$-MF type. To see this, let $X = (\varphi_1, \varphi_2, \ldots, \varphi_{d-1}) \in \mathrm{MF}_S^{d-1}(f)$ be indecomposable. Consider the d-fold factorization $\tilde{X} = (\varphi_1, \varphi_2, \ldots, \varphi_{d-1}, 1_{F_1}) \in \mathrm{MF}_S^d(f)$. To complete the proof, we show that $\tilde{X}$ is indecomposable in $\mathrm{MF}_S^d(f)$.

Suppose $\tilde{e} = (e_1, e_2, \ldots, e_d)$ is an idempotent in $\mathrm{End}_{\mathrm{MF}_S^d(f)}(\tilde{X})$. Then $e = (e_1, \ldots, e_{d-1})$ is an idempotent in $\mathrm{End}_{\mathrm{MF}_S^{d-1}(f)}(X)$. Since X is indecomposable by assumption, we have that $e = 0$ or $e = 1$. Since $\tilde{e} : \tilde{X} \to \tilde{X}$, it follows that $e_d = e_1$ and therefore, $\tilde{e} = 0$ or $\tilde{e} = 1$ implying that $\tilde{X}$ is indecomposable as well. □

In general, the converse of Lemma 4.3.3 does not hold (see Example 4.4.5).

4.3.1 The functors $(-)^\flat$ and $(-)^\sharp$

As in Section 4.1, let $\omega \in S$ be a primitive d-th root of 1 and $\mu \in S$ be any d-th root of -1. We start with a pair of functors between the categories $\mathrm{MCM}(R^\sharp)$ and $\mathrm{MF}_S^d(f)$ which are closely related to the functors $\mathcal{A}$ and $\mathcal{B}$ from Section 4.1 (see Lemma 4.3.8 for the precise relationship).

Definition 4.3.4.

(i) For $N \in \mathrm{MCM}(R^\sharp)$, let $\varphi : N \to N$ be the S-linear homomorphism representing multiplication by z on N and define

$$N^\flat = (\mu\varphi, \mu\varphi, \ldots, \mu\varphi) \in \mathrm{MF}_S^d(f).$$

For a homomorphism $g : N \to N'$ of $R^\sharp$-modules, define $g^\flat = (g, g, \ldots, g) \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(N^\flat, (N')^\flat)$.

(ii) For $X = (\varphi_1 : F_2 \to F_1, \varphi_2 : F_3 \to F_2, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$, define an MCM $R^\sharp$-module by setting $X^\sharp = \bigoplus_{k=0}^{d-1} F_{d-k}$ as an S-module with z-action given by:

$$z \cdot (x_d, x_{d-1}, \ldots, x_2, x_1) \coloneqq (\mu^{-1}\varphi_d(x_1), \mu^{-1}\varphi_{d-1}(x_d), \ldots, \mu^{-1}\varphi_1(x_2))$$

for all $x_i \in F_i, i \in \mathbb{Z}_d$. For a morphism $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) : X \to X'$, define $\alpha^\sharp = \bigoplus_{k=0}^{d-1} \alpha_{d-k} \in \mathrm{Hom}_{R^\sharp}(X^\sharp, (X')^\sharp)$.

As in Remark 4.1.4, the element μ allows for a factorization of f instead of $-f$. Moreover, the isomorphism classes of $N^\flat$ and $X^\sharp$ are not dependent on the root of $x^d + 1 \in S[x]$ chosen.

Recall the automorphism $\sigma : R^\sharp \to R^\sharp$ which fixes S and maps z to ωz. This automorphism acts on the category of MCM $R^\sharp$-modules in the following sense: For each $N \in \mathrm{MCM}(R^\sharp)$, let $(\sigma^k)^*N$ denote the MCM $R^\sharp$-module obtained by restricting scalars along $\sigma^k : R^\sharp \to R^\sharp$. Since $\sigma^d = 1_{R^\sharp}$, the mapping $N \mapsto \sigma^* N$ forms an autoequivalence of the category $\mathrm{MCM}(R^\sharp)$. We also recall the shift functor $T : \mathrm{MF}_S^d(f) \to \mathrm{MF}_S^d(f)$ given by $T(\varphi_1, \varphi_2, \ldots, \varphi_d) = (\varphi_2, \varphi_3, \ldots, \varphi_d, \varphi_1)$. It also gives an equivalence of $\mathrm{MF}_S^d(f)$ with itself satisfying $T^d = 1_{\mathrm{MF}_S^d(f)}$.

Proposition 4.3.5. *Let N be an MCM $R^\sharp$-module and $X \in \mathrm{MF}_S^d(f)$. Then*

$$X^{\sharp\flat} \cong \bigoplus_{k \in \mathbb{Z}_d} T^k(X) \quad \textit{and} \quad N^{\flat\sharp} \cong \bigoplus_{k \in \mathbb{Z}_d} (\sigma^k)^* N.$$

Proof. Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}_S^d(f)$. By definition $X^\sharp = F_d \oplus F_{d-1} \oplus \cdots \oplus F_1$ as an S-module and multiplication by z on $X^\sharp$ is given by $\mu^{-1}\varphi$ where

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & \varphi_d \\ \varphi_{d-1} & 0 & \cdots & 0 & 0 \\ 0 & \varphi_{d-2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_1 & 0 \end{pmatrix}.$$

Therefore, $X^{\sharp\flat} = (\varphi, \varphi, \ldots, \varphi) \in \mathrm{MF}_S^d(f)$. One can perform row and column operations to see that that $(\varphi, \varphi, \ldots, \varphi) \cong \bigoplus_{k \in \mathbb{Z}_d} T^k(X)$.

Notice that the first half of the proof is valid in any characteristic as long as there exists an element $\mu \in S$ satisfying $\mu^d = -1$. For instance, if d is odd then -1 is a valid choice. However, the second half of the proof explicitly makes use of the fact that $\mathrm{char}\,\mathbf{k}$ does not divide d.

In order to show the second isomorphism, let $N \in \mathrm{MCM}(R^\sharp)$ and let $\varphi : N \to N$ be the S-linear map representing multiplication by z on N. Then $N^\flat = (\mu\varphi : N \to N, \ldots, \mu\varphi : N \to N) \in \mathrm{MF}_S^d(f)$ and therefore $N^{\flat\sharp} = N \oplus N \oplus \cdots \oplus N$, the direct sum of d copies of the free S-module N. The z-action on $N^{\flat\sharp}$ is given by

$$\begin{aligned} z \cdot (n_d, n_{d-1}, \ldots, n_1) &= (\mu^{-1}\mu\varphi(n_1), \mu^{-1}\mu\varphi(n_d), \ldots, \mu^{-1}\mu\varphi(n_2)) \\ &= (zn_1, zn_d, \ldots, zn_2), \end{aligned}$$

for any $n_i \in N, i \in \mathbb{Z}_d$.

Let $k \in \mathbb{Z}_d$ and define a map $g_k : N^{\flat\sharp} \to (\sigma^k)^* N$ by mapping

$$n = (n_d, n_{d-1}, \ldots, n_1) \mapsto \frac{1}{d} \sum_{j=0}^{d-1} \omega^{jk} n_{d-j}$$

for any $n \in N^{\flat\sharp}$. Note that for $m \in (\sigma^k)^* N$, $z \cdot m = \omega^k zm$ by definition. Therefore, for

$n = (n_d, \dots, n_1) \in N^{\flat\sharp}$,

$$\begin{aligned}
z \cdot g_k(n) &= \frac{1}{d} \sum_{j=0}^{d-1} z \cdot (\omega^{jk} n_{d-j}) \\
&= \frac{1}{d} \sum_{j=0}^{d-1} \omega^{(j+1)k} z n_{d-j} \\
&= \frac{1}{d} \left(\omega^k z n_d + \omega^{2k} z n_{d-1} + \dots + \omega^{(d-1)k} z n_2 + z n_1 \right) \\
&= g_k(z \cdot n)
\end{aligned}$$

which implies that g_k is an $R^\sharp$-homomorphism. Putting these maps together we have an $R^\sharp$-homomorphism

$$g = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{d-1} \end{pmatrix} : N^{\flat\sharp} \to \bigoplus_{k=0}^{d-1} (\sigma^k)^* N.$$

In the other direction, we have $R^\sharp$-homomorphisms $s_k : (\sigma^k)^* N \to N^{\flat\sharp}$ given by

$$s_k(m) = (m, \omega^{-k} m, \omega^{-2k} m, \dots, \omega^{-(d-1)k} m)$$

for any $m \in (\sigma^k)^* N$. For each $k \in \mathbb{Z}_d$ and $m \in (\sigma^k)^* N$, we have that $g_k s_k(m) = m$. On the other hand, if $i \neq \ell \in \mathbb{Z}_d$, then

$$\begin{aligned}
g_i s_\ell(m) &= g_i(m, \omega^{-\ell} m, \omega^{-2\ell} m, \dots, \omega^{\ell} m) \\
&= \frac{1}{d} \sum_{j=0}^{d-1} \omega^{j(i-\ell)} m \\
&= 0.
\end{aligned}$$

Therefore, setting $s = \begin{pmatrix} s_0 & s_1 & \cdots & s_{d-1} \end{pmatrix}$, we have that

$$gs = \begin{pmatrix} g_0 s_0 & g_0 s_1 & \cdots & g_0 s_{d-1} \\ g_1 s_0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ g_{d-1} s_0 & \cdots & & g_{d-1} s_{d-1} \end{pmatrix}$$

which is the identity on $\bigoplus_{k \in \mathbb{Z}_d} (\sigma^k)^* N$. Hence g is a split surjection. However, since both the target and source of g have the same rank as free S-modules, we conclude that g is an isomorphism of $R^\sharp$-modules. $\square$

Corollary 4.3.6.

(i) For each $X \in \mathrm{MF}_S^d(f)$, there exists $N \in \mathrm{MCM}(R^\sharp)$ such that X is isomorphic to a summand of $N^\flat$.

(ii) For each $N \in \mathrm{MCM}(R^\sharp)$, there exists $X \in \mathrm{MF}_S^d(f)$ such that N is isomorphic to a summand of $X^\sharp$. $\square$

We can now state the main result of this section.

Theorem 4.3.7. *Let $d \geq 2$. Then f has finite d-MF type if and only if the d-fold branched cover $R^\sharp = S[\![z]\!]/(f + z^d)$ has finite CM type.*

The proof given below is lifted directly from the $d = 2$ case. Once again, the characteristic assumption on $\mathbf{k}$ is only needed in half of the proof as long there exists $\mu \in S$ satisfying $\mu^d = -1$.

Proof of Theorem 4.3.7. Let $X_1, X_2, \ldots, X_t$ be a representative list of the isomorphism classes of indecomposable d-fold matrix factorizations of f and let $N \in \mathrm{MCM}(R^\sharp)$ be indecomposable. Since $N^\flat \in \mathrm{MF}_S^d(f)$, there exist non-negative integers $s_1, s_2, \ldots, s_t$ such that

$N^\flat \cong X_1^{s_1} \oplus X_2^{s_2} \oplus \cdots \oplus X_t^{s_t}$. By Proposition 4.3.5, N is isomorphic to a direct summand of

$$N^{\flat\sharp} \cong (X_1^\sharp)^{s_1} \oplus (X_2^\sharp)^{s_2} \oplus \cdots \oplus (X_t^\sharp)^{s_t}.$$

Since N is indecomposable, KRS in $\mathrm{MCM}(R^\sharp)$ implies that N is isomorphic to a summand of $X_i^\sharp$ for some $1 \leq i \leq t$. Hence, every indecomposable MCM $R^\sharp$-module is isomorphic to one appearing in the finite list consisting of all summands of all $X_j^\sharp$, $1 \leq j \leq t$. The converse follows similarly from Proposition 4.3.5 and the KRS property in $\mathrm{MF}_S^d(f)$. □

In Section 4.1, it was shown the the category $\mathrm{MF}_S^d(f)$ is equivalent to the category of finitely generated modules over the skew group algebra $R^\sharp[\sigma]$ which are MCM as $R^\sharp$-modules. The equivalence is given by a pair of inverse functors $\mathcal{A} : \mathrm{MCM}_\sigma(R^\sharp) \to \mathrm{MF}_S^d(f)$ and $\mathcal{B} : \mathrm{MF}_S^d(f) \to \mathrm{MCM}_\sigma(R^\sharp)$. To finish this section, we make note of the connection between the functors $\mathcal{A}$ and $\mathcal{B}$ and the functors $(-)^\sharp$ and $(-)^\flat$.

Lemma 4.3.8. *Let $H : \mathrm{MCM}_\sigma(R^\sharp) \to \mathrm{MCM}(R^\sharp)$ be the functor which forgets the action of σ and $G : \mathrm{MCM}(R^\sharp) \to \mathrm{MCM}_\sigma(R^\sharp)$ be given by $G(N) = R^\sharp[\sigma] \otimes_{R^\sharp} N$ for any $N \in \mathrm{MCM}(R^\sharp)$.*

(i) For any $X \in \mathrm{MF}_S^d(f)$, $X^\sharp = H \circ \mathcal{B}(X)$.

(ii) For any $N \in \mathrm{MCM}(R^\sharp)$, $N^\flat \cong \mathcal{A} \circ G(N)$.

Proof. The first statement follows directly from the definition of $(-)^\sharp$ and $\mathcal{B}$. For the second, consider the idempotents

$$e_k = \frac{1}{d} \sum_{j \in \mathbb{Z}_d} \omega^{-jk} \sigma^j \in R^\sharp[\sigma], \quad k \in \mathbb{Z}_d.$$

These idempotents have three important properties:

(a) $R^\sharp[\sigma] = \bigoplus_{k \in \mathbb{Z}_d} e_k R^\sharp[\sigma]$ as right $R^\sharp[\sigma]$-modules,

(b) $\sigma e_k = e_k \sigma = \omega^k e_k$, $k \in \mathbb{Z}_d$, and

(c) $ze_k = e_{k-1}z$, $k \in \mathbb{Z}_d$.

From (b), we have that $e_k R^\sharp[\sigma] = e_k R^\sharp$ where $e_k R^\sharp$ denotes the multiples of e_k by $R^\sharp \cdot 1 \subset R^\sharp[\sigma]$ on the right. Hence, as an $R^\sharp$-module, $e_k R^\sharp$ is free of rank 1. Thus, for any $N \in \mathrm{MCM}(R^\sharp)$, (a) implies that $G(N) \cong \bigoplus_{k\in\mathbb{Z}_d}(e_k R^\sharp \otimes_{R^\sharp} N)$. It then follows from (b) and (c) that we have an isomorphism of $R^\sharp[\sigma]$-modules $G(N) \cong \mathcal{B}(N^\flat)$. Hence, $\mathcal{A} \circ G(N) \cong \mathcal{A} \circ \mathcal{B}(N^\flat) \cong N^\flat$ by Theorem 4.1.5. □

Remark 4.3.9. In the case of an Artin algebra Λ, the relationship between Λ and the skew group algebra $\Lambda[G]$ for a finite group G was studied by Reiten and Riedtmann in [RR85]. They show that many properties relevant to representation theory hold simultaneously for Λ and $\Lambda[G]$. In particular, Λ has finite representation type if and only if the same is true of $\Lambda[G]$. The equivalence of categories $\mathrm{MF}^d_S(f) \approx \mathrm{MCM}_\sigma(R^\sharp)$ and Theorem 4.3.7 give an analogous relationship between $R^\sharp$ and the skew group algebra $R^\sharp[\sigma]$.

4.4 Hypersurfaces of finite d-MF type

Let $(A, \mathfrak{m})$ be a regular local ring and $g \in \mathfrak{m}^2$ be non-zero. Then the hypersurface ring $A/(g)$ is called a *simple hypersurface singularity* if there are only finitely many proper ideals $I \subset A$ such that $g \in I^2$. In the case that A is a power series ring over an algebraically closed field of characteristic 0, the pair of papers [BGS87] and [Knö87] prove the following theorem.

Theorem 4.4.1 ([BGS87],[Knö87]). *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* 0 *and let* $R = \mathbf{k}[\![x_1, x_2, \ldots, x_r]\!]/(g)$*, where* $g \in (x_1, x_2, \ldots, x_r)^2$ *is non-zero. Then* R *has finite CM type if and only if* R *is a simple hypersurface singularity.*

Essential to their conclusion is the classification of simple hypersurface singularities, due to Arnol'd [Arn73], which gives explicit normal forms for all polynomials defining such a singularity. These are often referred to as the ADE singularities. The culmination of these results is a complete list of polynomials which define hypersurface rings of finite CM type in all dimensions (see [Yos90, Theorem 8.8] or [LW12, Theorem 9.8]). Equivalently, the polynomials in this list are precisely the ones with only finitely many indecomposable 2-fold matrix factorizations up to isomorphism.

Using Theorem 4.3.7 and the classification described above, we are able to compile a list of all f with finite d-MF type for $d > 2$.

Theorem 4.4.2. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* 0 *and* $S = \mathbf{k}[\![y, x_2, \ldots, x_r]\!]$*. Assume* $0 \neq f \in (y, x_2, \ldots, x_r)^2$ *and* $d > 2$*. Then* f *has finite* d-MF *type if and only if, after a possible change of variables,* f *and* d *are one of the following:*

(A_1)*:* $y^2 + x_2^2 + \cdots + x_r^2$ *for any* $d > 2$

(A_2)*:* $y^3 + x_2^2 + \cdots + x_r^2$ *for* $d = 3, 4, 5$

(A_3)*:* $y^4 + x_2^2 + \cdots + x_r^2$ *for* $d = 3$

(A_4)*:* $y^5 + x_2^2 + \cdots + x_r^2$ *for* $d = 3$

Proof. Let f and d be a pair in the list given. Then $f + z^d$ is a simple hypersurface singularity and therefore $R^\sharp$ has finite CM type by Theorem 4.4.1. By Theorem 4.3.7, f has finite d-MF type.

Conversely, let $0 \neq f \in (y, x_2, \ldots, x_r)^2$, $d > 2$, and assume f has finite d-MF type. Then $R^\sharp = S[\![z]\!]/(f + z^d)$ has dimension r and is of finite CM type by Theorem 4.3.7. We consider two cases.

First, assume $\dim R^\sharp = 1$, that is, assume $S = \mathbf{k}[\![y]\!]$. Then $f = uy^k$ for some unit $u \in S$ and $k \geq 2$. Since S is complete and $\operatorname{char} \mathbf{k} = 0$, there exists a k-th root v of u^{-1} in S [LW12, A.31]. Therefore, after replacing y with vy, we may assume that $f = y^k$. Since $\dim R^\sharp = 1$, [Yos90, p. 8.2.1] implies that $\operatorname{ord}(y^k + z^d) \leq 3$. Hence, either $k \leq 3$ or $d \leq 3$. If $k = 2$, there are no restrictions on d since $y^2 + z^d$ defines a simple (A_{d-1}) singularity for all $d > 2$. If $k = 3$, then the fact that $y^3 + z^d$ is a 1-dimensional simple hypersurface singularity implies that $d = 3, 4$, or 5. Similarly, if $d \leq 3$, then $d = 3$ and $k = 2, 3, 4$, or 5.

Next, assume $\dim R^\sharp \geq 2$. In this case, [Yos90, p. 8.2.2] implies that $\operatorname{ord}(f+z^d) \leq 2$. Since $d > 2$ and $f \in (y, x_2, \ldots, x_r)^2$, we have that $\operatorname{ord}(f) = 2$. By the Weierstrass Preparation Theorem [LW12, Corollary 9.6], there exists a unit $u \in S$ and $g \in \mathbf{k}[\![y, x_2, \ldots, x_{r-1}]\!]$ such that $f = (g + x_r^2)u$. As above, we may neglect the unit and assume that $f = g + x_r^2$ for some $g \in \mathbf{k}[\![y, x_2, \ldots, x_{d-1}]\!]$.

Since the hypersurface ring defined by $f + z^d = g + x_r^2 + z^d$ has finite CM type, Knörrer's theorem (Theorem 4.3.2) implies that $g + z^d$ defines a hypersurface ring of finite CM type as well. Thus, g has finite d-MF type by Theorem 4.3.7. We repeat this argument until $f = g' + x_2^2 + \cdots + x_r^2$ for some $g' \in \mathbf{k}[\![y]\!]$ with finite d-MF type. Finally, we apply the first case to g' to finish the proof. □

Corollary 4.4.3. *Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, $S = \mathbf{k}[\![y, x_2, \ldots, x_r]\!]$, and $f \in (y, x_2, \ldots, x_r)^2$ be non-zero. If f has finite d-MF type for some $d \geq 2$, then $R = S/(f)$ is an isolated singularity, that is, $R_\mathfrak{p}$ is a regular local ring for all non-maximal prime ideals $\mathfrak{p}$.*

Proof. The polynomials listed in Theorem 4.4.2 are a subset of the ones in [Yos90, Theorem 8.8] (or [LW12, Theorem 9.8]), all of which define isolated singularities. □

Suppose we have a pair f and d from the list in Theorem 4.4.2 such that $R^\sharp$ has dimension 1. Then [Yos90, Chapter 9] gives matrix factorizations for every indecomposable MCM $R^\sharp$-module. By computing multiplication by z on each of the corresponding $R^\sharp$-modules, we can compile a representative list of all isomorphism classes of indecomposable d-fold factorizations of f. We give one such computation in the following example.

Example 4.4.4. Let $\mathbf{k}$ be algebraically closed of characteristic 0. Let $S = \mathbf{k}[\![y]\!]$, $f = y^4 \in S$, and $R = S/(f)$. The hypersurface ring $R^\sharp = \mathbf{k}[\![x, y]\!]/(y^4 + x^3)$ is a simple curve singularity of type E_6 and has finite CM type. Here we are viewing $R^\sharp$ as the 3-fold branched cover of R. By Theorem 4.3.7, the category $\mathrm{MF}_S^3(y^4)$ has only finitely many non-isomorphic indecomposable objects. We give a complete list below.

A complete list of non-isomorphic indecomposable MCM $R^\sharp$-modules is given in [Yos90, p. 9.13]. By Corollary 4.3.6, we may compute multiplication by x on each of these modules to obtain a representative from each isomorphism class of indecomposable matrix factorizations of y^4 with 3 factors. By Remark 4.1.4, we may choose $\mu = -1$.

Following the notation of [Yos90, p. 9.13], we let $\varphi_1 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}$ and $M_1 = \operatorname{cok} \varphi_1$. Let e_1 and e_2 in M_1 denote the images of the standard basis on $S[\![x]\!]^2$. Then e_1 and e_2 satisfy $xe_1 = -y^3 e_2$ and $x^2 e_2 = y e_1$. As an S-module, M_1 is free with basis $\{e_1, e_2, xe_2\}$. Multiplication by x on M_1 is therefore given by

$$\varphi = \begin{pmatrix} & & y \\ -y^3 & & \\ & 1 & \end{pmatrix}.$$

Hence, $M_1^\flat = (-\varphi, -\varphi, -\varphi) \in \mathrm{MF}_S^3(y^4)$. Furthermore, we have a commutative diagram

$$\begin{array}{ccccccc}
S^3 & \xrightarrow{-\varphi} & S^3 & \xrightarrow{-\varphi} & S^3 & \xrightarrow{-\varphi} & S^3 \\
\Big\downarrow {\scriptstyle \begin{pmatrix} & 1 & \\ -1 & & \\ & & -1 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} & & -1 \\ & -1 & \\ 1 & & \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} & 1 & \\ -1 & & \\ & & -1 \end{pmatrix}} \\
S^3 & \xrightarrow[{\scriptstyle \begin{pmatrix} 1 & & \\ & y^3 & \\ & & y \end{pmatrix}}]{} & S^3 & \xrightarrow[{\scriptstyle \begin{pmatrix} y & & \\ & 1 & \\ & & y^3 \end{pmatrix}}]{} & S^3 & \xrightarrow[{\scriptstyle \begin{pmatrix} y^3 & & \\ & y & \\ & & 1 \end{pmatrix}}]{} & S^3.
\end{array}$$

Thus, $M_1^\flat$ is isomorphic to the direct sum of the indecomposable factorization $X_{\varphi_1} := (y^3, y, 1)$ and its corresponding shifts, that is, $M_1^\flat \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(y^3, y, 1)$.

Similarly, multiplication by x can be computed for each of the indecomposable MCM $R^\sharp$ modules listed in [Yos90, p. 9.13]. From this computation, we obtain the list of indecomposable 3-fold matrix factorizations of y^4 given in Table 4.1.

The factorizations $\mathcal{P}_1, X_{\varphi_1}, X_{\psi_1}, X_{\varphi_2}$, and X_β are each indecomposable since they are of size 1. By Corollary 2.3.12, the cosyzygy of an indecomposable reduced matrix factorization

Table 4.1: Indecomposable objects in $\mathrm{MF}^3_{\mathbf{k}[\![y]\!]}(y^4)$

$X \in \mathrm{MF}^3_S(y^4)$	$N^\flat$ for $N \in \mathrm{MCM}(R^\sharp)$
$\mathcal{P}_1 = (y^4, 1, 1)$	$(R^\sharp)^\flat \cong \bigoplus_{i\in\mathbb{Z}_3} T^i(\mathcal{P}_1)$
$X_{\varphi_1} = (y^3, y, 1)$	$M_1^\flat \cong \bigoplus_{i\in\mathbb{Z}_3} T^i(X_{\varphi_1})$
$X_{\psi_1} = (y^3, 1, y)$	$N_1^\flat \cong \bigoplus_{i\in\mathbb{Z}_3} T^i(X_{\psi_1})$
$X_{\varphi_2} = (y^2, y^2, 1)$	$M_2^\flat \cong N_2^\flat \cong \bigoplus_{i\in\mathbb{Z}_3} T^i(X_{\varphi_2})$
$X_\beta = (y^2, y, y)$	$B^\flat \cong \bigoplus_{i\in\mathbb{Z}_3} T^i(X_\beta)$
$X_\alpha = \left(\begin{pmatrix} 0 & -y^2 \\ 1 & -y \end{pmatrix}, \begin{pmatrix} 0 & -y^3 \\ 1 & -y^2 \end{pmatrix}, \begin{pmatrix} 0 & -y^3 \\ 1 & -y \end{pmatrix}\right)$	$A^\flat \cong \bigoplus_{i\in\mathbb{Z}_3} T^i(X_\alpha)$
$X_\xi = \left(\begin{pmatrix} y & 0 \\ 0 & y^3 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix}, \begin{pmatrix} -y & 1 \\ y^2 & 0 \end{pmatrix}\right)$	$X^\flat \cong \bigoplus_{i\in\mathbb{Z}_3} T^i(X_\xi)$.

is again indecomposable. Here a reduced matrix factorization means all the entries of all the matrices lie in the maximal ideal of S (see Section 4.6). Using (2.2.10), we have that $\Omega^-_{\mathrm{MF}^3_S(y^4)}(X_\beta) \cong X_\alpha$ and therefore, X_α is indecomposable.

Since X_ξ is of size 2, a non-trivial decomposition would be of the form $(y, b, c) \oplus (y^3, b', c')$ for some $b, c, b', c' \in S$. Since $\det \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix} = -y^2$, the possibilities for b and b' are, up to units, $b = y^2$ and $b' = 1$ or $b = y = b'$. By considering cokernels, both cases lead to contradictions and so X_ξ must be indecomposable.

By Proposition 4.3.5, these seven factorizations, and each of their corresponding shifts, give the complete list of non-isomorphic indecomposable objects in $\mathrm{MF}^3_S(y^4)$ (21 in total).

To end this section, we discuss the relationship between the 2-MF type of f and the d-MF type of f for $d > 2$. As we saw in Lemma 4.3.3, finite d-MF type implies finite 2-MF type. The example below shows that the converse does not hold in general. In particular, we give a polynomial of finite 2-MF type which has infinite 3-MF type.

Example 4.4.5. Let $S = \mathbf{k}[\![x,y]\!]$ where $\mathbf{k}$ is an algebraically closed field with $\operatorname{char}\mathbf{k} \neq 2,3$ and let $f = x^3 + y^3 \in S$. The hypersurface ring $R = S/(f)$ is a simple singularity of type D_4 and therefore has finite CM type.

Consider $R^\sharp = \mathbf{k}[\![x,y,z]\!]/(x^3+y^3+z^3)$, the 3-fold branched cover of R. Following [BP15], to each point $(a,b,c) \in \mathbf{k}^3$ satisfying $a^3+b^3+c^3 = 0$ and $abc \neq 0$, we associate the *Moore matrix*

$$X_{abc} = \begin{pmatrix} ax & bz & cy \\ by & cx & az \\ cz & ay & bx \end{pmatrix}.$$

The cokernel $N_{abc} = \operatorname{cok}(X_{abc})$ is an MCM $R^\sharp$-module and is given by the matrix factorization $(X_{abc}, \frac{1}{abc}\operatorname{adj}X_{abc}) \in \operatorname{MF}^2_{S[\![z]\!]}(x^3+y^3+z^3)$, where $\operatorname{adj}X_{abc}$ is the classical adjoint of X_{abc}. Furthermore, N_{abc} is indecomposable since $\det X_{abc} = abc(x^3+y^3+z^3)$ and $x^3+y^3+z^3$ is irreducible. Buchweitz and Pavlov give precise conditions for X_{abc} to be matrix equivalent to $X_{a'b'c'}$ (see [BP15, Proposition 2.13]). In particular, their results imply that the collection $\{N_{abc}\}$, as (a,b,c) varies over the curve $x^3+y^3+z^3$, gives an uncountable collection of non-isomorphic indecomposable MCM $R^\sharp$-modules.

With respect to the images of the standard basis on $S[\![z]\!]^3$, multiplication by z on N_{abc} is given by the S-matrix

$$\varphi_{abc} = \begin{pmatrix} 0 & -\frac{c}{a}y & -\frac{a}{c}x \\ -\frac{c}{b}x & 0 & -\frac{b}{c}y \\ -\frac{a}{b}y & -\frac{b}{a}x & 0 \end{pmatrix}.$$

Therefore, we have that $N^\flat_{abc} = (\mu\varphi_{abc}, \mu\varphi_{abc}, \mu\varphi_{abc}) \in \operatorname{MF}^3_S(x^3+y^3)$, where $\mu^3 = -1$. For any $N_1, N_2 \in \operatorname{MCM}(R^\sharp)$, $N_1^\flat \cong N_2^\flat$ if and only if $N_1 \cong (\sigma^k)^* N_2$ for some $k \in \mathbb{Z}_d$. Hence, the collection of non-isomorphic indecomposable summands of $N^\flat_{abc}$ for all (a,b,c) as above cannot be a finite set. It follows that x^3+y^3 has infinite 3-MF type. Furthermore, the entries of φ_{abc} lie in the maximal ideal of S so x^3+y^3 has infinite reduced 3-MF type as well (see Section 4.6).

4.5 Decomposability of $N^\flat$ and $X^\sharp$

Let $d \geq 2$ and $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring. Assume $\mathbf{k}$ is algebraically closed of characteristic not dividing d. Let $f \in \mathfrak{n}^2$ be non-zero, $R = S/(f)$, and $R^\sharp = S[\![z]\!]/(f + z^d)$. Proposition 4.3.5 showed that both $N^{\flat\sharp}$ and $X^{\sharp\flat}$ decompose into a sum of d objects. In this section we investigate the decomposability of $N^\flat$ and $X^\sharp$.

Recall that the shift functor $T : \mathrm{MF}_S^d(f) \to \mathrm{MF}_S^d(f)$ satisfies $T^d = 1_{\mathrm{MF}_S^d(f)}$. In particular, for any $X \in \mathrm{MF}_S^d(f)$, there exists a smallest integer $k \in \{1, 2, \ldots, d-1, d\}$ such that $T^k X \cong X$. We call k the *order* of X.

Lemma 4.5.1. *For any $X \in \mathrm{MF}_S^d(f)$, the order of X is a divisor of d.*

Proof. For a given $X \in \mathrm{MF}_S^d(f)$, the cyclic group of order d generated by T acts on the set of equivalence classes $\{[T^i X] : i \in \mathbb{Z}_d\}$. In particular, the stabilizer of $[X]$ is generated by T^k for some $k \mid d$ which can be taken to be the smallest possible in $\{1, 2, \ldots, d\}$. It follows that the order of X is k. □

The next result builds on an idea of Knörrer [Knö87, Lemma 1.3] and Gabriel [Gab81, p. 95]. The proof is based on [LW12, Lemma 8.25] which states that a matrix factorization $(\varphi, \psi) \in \mathrm{MF}_S^2(f)$ satisfying $(\varphi, \psi) \cong (\psi, \varphi)$ is isomorphic to a factorization of the form (φ_0, φ_0). For $d > 2$, the situation is similar, but the divisors of d play a role. Specifically, if X has order k, then X is isomorphic to the concatenation of k matrices, d/k times.

Proposition 4.5.2. *Let $X \in \mathrm{MF}_S^d(f)$ be indecomposable of size n and assume X has order $k < d$. Then there exist S-homomorphisms $\varphi_1', \varphi_2', \ldots, \varphi_k'$ such that $(\varphi_1'\varphi_2'\cdots\varphi_k')^{\frac{d}{k}} = f \cdot I_n$ and*

$$X \cong (\varphi_1', \varphi_2', \ldots, \varphi_k', \varphi_1', \varphi_2', \ldots, \varphi_k', \ldots, \varphi_1', \varphi_2', \ldots, \varphi_k').$$

Proof. Let $X = (\varphi_1 : F_2 \to F_1, \varphi_2 : F_3 \to F_2, \ldots, \varphi_d : F_1 \to F_d)$ and set $r = d/k$. By assumption, there is an isomorphism $\alpha = (\alpha_1, \ldots, \alpha_d) : X \to T^k X$. By applying $T^k(-)$

repeatedly, we obtain an automorphism $\tilde{\alpha}$ of X defined by the composition

$$X \xrightarrow{\alpha} T^k X \xrightarrow{T^k(\alpha)} T^{2k} X \xrightarrow{T^{2k}(\alpha)} \cdots \xrightarrow{T^{(r-2)k}(\alpha)} T^{(r-1)k} X \xrightarrow{T^{(r-1)k}(\alpha)} X.$$

In particular, $\tilde{\alpha} = (\alpha_{i+(r-1)k}\alpha_{i+(r-2)k} \cdots \alpha_{i+k}\alpha_i)_{i=1}^d$. Since X is indecomposable the endomorphism ring $\Lambda \coloneqq \operatorname{End}_{\mathrm{MF}_S^d(f)}(X)$ is local. Since $\mathbf{k}$ is algebraically closed, it cannot have any non-trivial finite extensions which are division rings. Hence, the division ring $\Lambda/\operatorname{rad}\Lambda$ must be isomorphic to $\mathbf{k}$. This allows us to write

$$\tilde{\alpha} = c \cdot 1_X + \rho$$

for some $c \in \mathbf{k}^\times$ and $\rho \in \operatorname{rad}\Lambda$. Since $\operatorname{char}\mathbf{k} \nmid d$, we may scale α by $c^{-\frac{1}{r}}$ and assume $\tilde{\alpha} = 1_X + \rho$ for $\rho = (\rho_1, \rho_2, \ldots, \rho_d) \in \operatorname{rad}\Lambda$.

If $i \in \mathbb{Z}_d$, then

$$\begin{aligned} \alpha_i\rho_i &= \alpha_i(\alpha_{i+(r-1)k}\alpha_{i+(r-2)k} \cdots \alpha_{i+k}\alpha_i - 1_{F_i}) \\ &= (\alpha_i\alpha_{i+(r-1)k}\alpha_{i+(r-2)k} \cdots \alpha_{i+k} - 1_{F_{i+k}})\alpha_i \\ &= \rho_{i+k}\alpha_i. \end{aligned}$$

Represent the function $g(x) = (1+x)^{-1/r}$ by its Maclaurin series and define, for each $i \in \mathbb{Z}_d$,

$$\beta_i \coloneqq \alpha_i g(\rho_i) = g(\rho_{i+k})\alpha_i : F_i \to F_{i+k}.$$

For $i \in \mathbb{Z}_d$, we have that

$$\beta_i\varphi_i = g(\rho_{i+k})\alpha_i\varphi_i = g(\rho_{i+k})\varphi_{i+k}\alpha_{i+1} = \varphi_{i+k}g(\rho_{i+k+1})\alpha_{i+1} = \varphi_{i+k}\beta_{i+1}.$$

Hence, $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \operatorname{Hom}_{\mathrm{MF}_S^d(f)}(X, T^k X)$. By repeatedly applying $\alpha_i g(\rho_i) =$

$g(\rho_{i+k})\alpha_i$, we have that

$$\begin{aligned}
\beta_i\beta_{i-k}\beta_{i-2k}\cdots\beta_{i+2k}\beta_{i+k} &= (\alpha_i g(\rho_i))(\alpha_{i-k}g(\rho_{i-k}))(\alpha_{i-2k}g(\rho_{i-2k}))\cdots(\alpha_{i+k}g(\rho_{i+k})) \\
&= \alpha_i\alpha_{i-k}\cdots\alpha_{i+2k}\alpha_{i+k}g(\rho_{i+k})^r \\
&= (1_{F_{i+k}} + \rho_{i+k})(1_{F_{i+k}} + \rho_{i+k})^{-1} \\
&= 1_{F_{i+k}}.
\end{aligned}$$

Hence, β_i is an isomorphism for each $i \in \mathbb{Z}_d$ and therefore the morphism β is an isomorphism of matrix factorizations.

We claim that $X \cong (\beta_1\varphi_1, \ldots, \varphi_k, \beta_1\varphi_1, \ldots, \varphi_k, \ldots, \beta_1\varphi_1, \ldots, \varphi_k)$. For $0 \leq j \leq r-1$ and $2 \leq t \leq k+1$, define $\gamma_{j,t}$ to be the composition of the homomorphisms β_i beginning at F_{t+jk} of length $r-j$. In other words,

$$\gamma_{j,k} = \beta_{t+(r-1)k}\beta_{t+(r-2)k}\cdots\beta_{t+(j+1)k}\beta_{t+jk} : F_{t+jk} \to F_t.$$

Note that each $\gamma_{j,k}$ is an isomorphism. For $j = 0$, the index -1 is interpreted as $r-1$ so that $\gamma_{-1,k+1} = \beta_1$.

Let $0 \leq j \leq r-1$ and $2 \leq t \leq k+1$. To finish the proof, it suffices to show that the following diagram commutes:

$$\begin{array}{ccccccccc}
F_{k+1+jk} & \xrightarrow{\varphi_{k+jk}} & F_{k+jk} & \xrightarrow{\varphi_{k-1+jk}} & \cdots & \xrightarrow{\varphi_{2+jk}} & F_{2+jk} & \xrightarrow{\varphi_{1+jk}} & F_{1+jk} \\
\downarrow{\scriptstyle \gamma_{j,k+1}} & & \downarrow{\scriptstyle \gamma_{j,k}} & & & & \downarrow{\scriptstyle \gamma_{j,2}} & & \downarrow{\scriptstyle \gamma_{j-1,k+1}} \\
F_{k+1} & \xrightarrow{\varphi_k} & F_k & \xrightarrow{\varphi_{k-1}} & \cdots & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\beta_1\varphi_1} & F_{k+1}.
\end{array}$$

The commutativity can be broken into three steps. First, we show that $\gamma_{j-1,k+1}\varphi_{1+jk} = \beta_1\varphi_1\gamma_{j,2}$.

By repeatedly applying $\beta_i\varphi_i = \varphi_{i+k}\beta_{i+1}$, $i \in \mathbb{Z}_d$, we have that

$$\begin{aligned}\gamma_{j-1,k+1}\varphi_{1+jk} &= \beta_1\beta_{1-k}\beta_{1-2k}\cdots\beta_{1+(j+1)k}\beta_{1+jk}\varphi_{1+jk}\\ &= \beta_1\varphi_1\beta_{2-k}\beta_{2-2k}\cdots\beta_{2+(j+1)k}\beta_{2+jk}\\ &= \beta_1\varphi_1\gamma_{j,2}.\end{aligned}$$

Similarly, for $2 \leq t \leq k$, we have that

$$\begin{aligned}\gamma_{j,t}\varphi_{t+jk} &= \beta_{t-k}\beta_{t-2k}\cdots\beta_{t+(j+1)k}\beta_{t+jk}\varphi_{t+jk}\\ &= \varphi_t\beta_{t+1-k}\beta_{t+1-2k}\cdots\beta_{t+1+(j+1)k}\beta_{t+1+jk}\\ &= \varphi_t\gamma_{j,t+1}\end{aligned}$$

and

$$\begin{aligned}\gamma_{j,k}\varphi_{k+jk} &= \beta_d\beta_{-k}\beta_{-2k}\cdots\beta_{(j+1)k}\varphi_{(j+1)k}\\ &= \varphi_k\beta_1\beta_{1-k}\cdots\beta_{1+(j+1)k}\\ &= \varphi_k\gamma_{j,k+1}.\end{aligned}$$

Thus, the d-tuple $\gamma = (\gamma_{-1,k+1}, \gamma_{0,2}, \gamma_{0,3}, \ldots, \gamma_{0,k+1}, \gamma_{1,2}, \ldots, \gamma_{r-1,k})$ forms an isomorphism from X to $(\beta_1\varphi_1, \ldots, \varphi_k, \beta_1\varphi_1, \ldots, \varphi_k, \ldots, \beta_1\varphi_1, \ldots, \varphi_k)$. □

Example 4.5.3. Let $X = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathrm{MF}_S^4(f)$ and assume X has order 4. Then $Y = X \oplus T^2X$ satisfies $T^2Y \cong Y$ but $TY \not\cong Y$. Hence, Y has order 2. As in Proposition 4.5.2, we have an isomorphism

$$\begin{aligned}Y &= \left(\begin{pmatrix}\varphi_1 & \\ & \varphi_3\end{pmatrix}, \begin{pmatrix}\varphi_2 & \\ & \varphi_4\end{pmatrix}, \begin{pmatrix}\varphi_3 & \\ & \varphi_1\end{pmatrix}, \begin{pmatrix}\varphi_4 & \\ & \varphi_2\end{pmatrix}\right)\\ &\cong \left(\begin{pmatrix} & \varphi_3\\ \varphi_1 & \end{pmatrix}, \begin{pmatrix}\varphi_2 & \\ & \varphi_4\end{pmatrix}, \begin{pmatrix} & \varphi_3\\ \varphi_1 & \end{pmatrix}, \begin{pmatrix}\varphi_2 & \\ & \varphi_4\end{pmatrix}\right).\end{aligned}$$

So, Y is isomorphic to the concatenation of the two matrices $\begin{pmatrix} & \varphi_3 \\ \varphi_1 & \end{pmatrix}$ and $\begin{pmatrix} \varphi_2 & \\ & \varphi_4 \end{pmatrix}$.

The special case of order 1 will be important going forward.

Corollary 4.5.4. *Let $X \in \mathrm{MF}_S^d(f)$ be indecomposable of size n and assume that $X \cong TX$. Then there exists a homomorphism $\varphi : S^n \to S^n$ such that $\varphi^d = f \cdot I_n$ and $X \cong (\varphi, \varphi, \ldots, \varphi)$.* □

Proposition 4.5.5. *Let $X \in \mathrm{MF}_S^d(f)$, $N \in \mathrm{MCM}(R^\sharp)$, and assume both X and N are indecomposable objects.*

(i) If $X \cong TX$, then $X \cong M^\flat$ for some $M \in \mathrm{MCM}(R^\sharp)$.

(ii) If $N \cong \sigma^* N$, then $N \cong Y^\sharp$ for some $Y \in \mathrm{MF}_S^d(f)$.

Proof. If $X \cong TX$, then Corollary 4.5.4 implies that there exists a free S-module F and an endomorphism $\varphi : F \to F$ such that $\varphi^d = f \cdot 1_F$ and $X \cong (\varphi, \varphi, \ldots, \varphi) \in \mathrm{MF}_S^d(f)$. The pair $(F, \mu^{-1}\varphi)$ defines an $\mathrm{MCM}(R^\sharp)$ module M as follows: As an S-module, $M = F$, and the z-action on M is given by $z \cdot m = \mu^{-1}\varphi(m)$ for all $m \in M$, where $\mu \in S$ satisfies $\mu^d = -1$. Since $(\mu^{-1}\varphi)^d = -f \cdot 1_M$, M is naturally an $R^\sharp$-module. Since $M = F$ is free over S, it is MCM over $R^\sharp$. By applying $(-)^\flat$, we have that $M^\flat = (\varphi, \varphi, \ldots, \varphi) \cong X$.

Assume $N \cong \sigma^* N$. Using a similar technique to the proof of Proposition 4.5.2, we obtain an isomorphism of $R^\sharp$-modules $\theta : N \to \sigma^* N$ such that

$$(\sigma^{d-1})^*\theta \circ (\sigma^{d-2})^*\theta \circ \cdots \circ \sigma^*\theta \circ \theta = 1_N.$$

Such an isomorphism defines the structure of an $R^\sharp[\sigma]$-module on N. Thus, by Theorem 4.1.5, there exists $Y \in \mathrm{MF}_S^d(f)$ such that $\mathcal{B}(Y) \cong N$ as $R^\sharp[\sigma]$-modules and therefore $Y^\sharp \cong N$ as $R^\sharp$-modules by Lemma 4.3.8(i). □

Proposition 4.5.6. *Let X be indecomposable in $\mathrm{MF}_S^d(f)$ and N be indecomposable in $\mathrm{MCM}(R^\sharp)$.*

(i) Assume $X \cong TX$*. Then* $X^\sharp \cong \bigoplus_{k\in\mathbb{Z}_d}(\sigma^k)^*M$ *for some indecomposable* $M \in \mathrm{MCM}(R^\sharp)$ *such that* $M \not\cong \sigma^*M$*.*

(ii) The number of indecomposable summands of $X^\sharp$ *is at most* d*. Furthermore, if* $X^\sharp$ *has exactly* d *indecomposable summands, then* $X \cong TX$*.*

(iii) Assume $N \cong \sigma^*N$*. Then* $N^\flat \cong \bigoplus_{k\in\mathbb{Z}_d} T^kY$ *for some indecomposable* $Y \in \mathrm{MF}^d_S(f)$ *such that* $Y \not\cong TY$*.*

(iv) The number of indecomposable summands of $N^\flat$ *is at most* d*. Furthermore, if* $N^\flat$ *has exactly* d *indecomposable summands, then* $N \cong \sigma^*N$*.*

Proof. If $X \cong TX$, then Proposition 4.5.5(i) implies that $X \cong M^\flat$ for some $M \in \mathrm{MCM}(R^\sharp)$. By Proposition 4.3.5, we have that $X^\sharp \cong M^{\flat\sharp} \cong \bigoplus_{k\in\mathbb{Z}_d}(\sigma^k)^*M$. Similarly, if $N \cong \sigma^*N$, then Proposistion 4.5.5(ii) and Proposition 4.3.5 imply that $N^\flat \cong \bigoplus_{k\in\mathbb{Z}_d} T^kY$ for some $Y \in \mathrm{MF}^d_S(f)$.

Next, in the case that $TX \cong X$, we show that M above is indecomposable and satisfies $M \not\cong \sigma^*M$.

- Suppose $M \cong M_1 \oplus M_2$ for non-zero $M_1, M_2 \in \mathrm{MCM}(R^\sharp)$. Then $(\sigma^k)^*M \cong (\sigma^k)^*M_1 \oplus (\sigma^k)^*M_2$ for each $k \in \mathbb{Z}_d$. Therefore,

$$X^d \cong X^{\sharp\flat} \cong \bigoplus_{k\in\mathbb{Z}_d}((\sigma^k)^*M_1)^\flat \oplus ((\sigma^k)^*M_2)^\flat.$$

  This contradicts KRS since the left side has precisely d indecomposable summands while the right hand side has at least $2d$ indecomposable summands. Hence, M is indecomposable.

- Suppose that $\sigma^*M \cong M$. Then, since M is indecomposable, the arguments above imply that $M^\flat$ decomposes into a sum of at least d indecomposable summands. Since $T(M^\flat) \cong M^\flat$, we have

$$X^d \cong X^{\sharp\flat} \cong (M^\flat)^{\sharp\flat} \cong (M^\flat)^d.$$

Since X is indecomposable, the left hand side has precisely d indecomposable summands while the right hand side has at least d^2 indecomposable summands. Once again, we have a contradiction and so $M \ncong \sigma^* M$.

This completes the proof of (i). We omit the remaining assertions from (iii) as they follow similarly.

In order to prove (ii), suppose $X^\sharp = M_1 \oplus M_2 \oplus \cdots \oplus M_t$ for non-zero $M_i \in \mathrm{MCM}(R^\sharp)$. Then

$$X \oplus TX \oplus \cdots \oplus T^{d-1}X \cong X^{\sharp\flat} \cong M_1^\flat \oplus M_2^\flat \oplus \cdots \oplus M_t^\flat. \tag{4.5.1}$$

The left hand side has precisely d indecomposable summands and therefore $t \leq d$.

If $X^\sharp$ decomposes into exactly d indecomposables, that is, if $t = d$, then (4.5.1) implies that $M_i^\flat$ is indecomposable for each i and that $X \cong M_j^\flat$ for some $1 \leq j \leq d$. Then $TX \cong T(M_j^\flat) = M_j^\flat \cong X$.

The proof of (iv) is similar, observing that $\sigma^*(X^\sharp) \cong X^\sharp$ for any $X \in \mathrm{MF}_S^d(f)$.

□

4.6 Reduced matrix factorizations

Let $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring, $0 \neq f \in \mathfrak{n}^2$, and let $d \geq 2$ be an integer. Assume $\mathbf{k}$ is algebraically closed of characteristic not dividing d. In this section, we will consider the following special class of matrix factorizations in $\mathrm{MF}_S^d(f)$.

Definition 4.6.1. A matrix factorization $X = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in \mathrm{MF}_S^d(f)$ is called *reduced* if $\varphi_k : F_{k+1} \to F_k$ is minimal for each $k \in \mathbb{Z}_d$, that is, if $\operatorname{Im} \varphi_k \subseteq \mathfrak{n} F_k$. Equivalently, after choosing bases, X is reduced if the entries of φ_k lie in $\mathfrak{n}$ for all $k \in \mathbb{Z}_d$. We say that f has *finite reduced d-MF type* if there are, up to isomorphism, only finitely many indecomposable reduced matrix factorizations $X \in \mathrm{MF}_S^d(f)$.

In the case $d = 2$, any indecomposable non-reduced matrix factorization is isomorphic to either $(1, f)$ or $(f, 1)$ in $\mathrm{MF}_S^2(f)$ [Yos90, Remark 7.5]. In particular, this implies that finite 2-MF type is equivalent to finite reduced 2-MF type.

For $d > 2$, the situation is quite different. There at least as many non-reduced indecomposable d-fold factorizations of f as there are reduced ones (see Corollary 2.3.12). Moreover, finite d-MF type clearly implies finite reduced d-MF type but the converse does not hold for $d > 2$ as we will show in Example 4.6.11.

Definition 4.6.2.

(i) Let $X = (\varphi_1, \dots, \varphi_d) \in \mathrm{MF}_S^d(f)$ and pick bases to consider φ_k, $k \in \mathbb{Z}_d$, as a square matrix with entries in S. Following [BGS87], we define $I(\varphi_k)$ to be the ideal generated by the entries of φ_k and set $I(X) = \sum_{k \in \mathbb{Z}_d} I(\varphi_k)$. Note that the ideal $I(X)$ does not depend on the choice of basis.

(ii) Let $c_d(f)$ denote the collection of proper ideals I of S such that $f \in I^d$.

In the case $d = 2$, Theorem 4.4.1 implies that the reduced 2-MF type of f is determined by the cardinality of the set $c_2(f)$. One implication of Theorem 4.4.1 is proven explicitly in [BGS87]. The authors show that the association $X \mapsto I(X)$ forms a surjection from the set of isomorphism classes of reduced 2-fold matrix factorizations of f onto the set $c_2(f)$. Hence, if there are only finitely many indecomposable reduced 2-fold matrix factorizations of f up to isomorphism, then the set $c_2(f)$ is finite.

The following result of Herzog, Ulrich, and Backelin shows that the association $X \mapsto I(X)$ remains surjective in the case $d > 2$.

Theorem 4.6.3 ([HUB91], Theorem 1.2)**.** *Let I be a proper ideal of S and $d \geq 2$. If $f \in I^d$, then there exists a reduced matrix factorization $X \in \mathrm{MF}_S^d(f)$ such that $I(X) = I$.* □

Corollary 4.6.4. *Suppose f has finite reduced d-MF type. Then $c_d(f)$ is a finite collection of ideals of S.* □

Corollary 4.6.4 extends one direction of Theorem 4.4.1; however, the converse does not hold for $d > 2$ as shown by the next example.

Example 4.6.5. Let $S = \mathbf{k}[\![x,y]\!]$ where $\mathbf{k}$ algebraically closed with $\operatorname{char} \mathbf{k} \neq 2$ and $f = x^2y \in S$. Then the one-dimensional D_∞ singularity $R = S/(f)$ has countably infinite CM type by [BGS87, Proposition 4.2]. For each $k \geq 1$, we have a reduced matrix factorization of x^2y with 3 factors:

$$X_k = \left(\begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix} \right) \in \mathrm{MF}_S^3(x^2y).$$

Any isomorphism $X_k \to X_j$ for $k, j \geq 1$ induces an isomorphism of R-modules $\operatorname{cok} \begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix} \to \operatorname{cok} \begin{pmatrix} x & y^j \\ 0 & -x \end{pmatrix}$. Such an isomorphism is only possible if $k = j$, that is, $X_k \cong X_j$ if and only if $k = j$. Since X_k is reduced and the MCM R-module $\operatorname{cok} \begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix}$ is indecomposable, Corollary 2.3.12 implies that X_k is indecomposable. Thus, x^2y has infinite reduced d-MF type.

On the other hand, we claim that $c_3(x^2y)$ contains only the maximal ideal. To see this, suppose I is a proper ideal of S such that $f = x^2y \in I^3$. Notice that if $a \in I^3$, then $\frac{\partial}{\partial x}(a) \in I^2$ (and $\frac{\partial}{\partial y}(a) \in I^2$). Hence, $\frac{\partial}{\partial x}(x^2y) = 2xy \in I^2$. Similarly, $2xy \in I^2$ implies that $\frac{\partial}{\partial x}(2xy) = 2y \in I$ and $\frac{\partial}{\partial y}(2xy) = 2x \in I$. It follows that $I = (x,y)$ and $c_3(x^2y) = \{(x,y)\}$. So, $c_3(x^2y)$ is finite but x^2y has infinite reduced 3-MF type.

4.6.1 Ulrich modules and reduced matrix factorizations

Let N be an MCM $R^\sharp$-module and let $\mu_{R^\sharp}(N)$ denote the size of a minimal generating set of N. Recall that N is finitely generated and free over S. We will see below that there is an inequality

$$\mu_{R^\sharp}(N) \leq \operatorname{rank}_S(N). \tag{4.6.1}$$

In the following, we consider MCM $R^\sharp$-modules N where the equality $\mu_{R^\sharp}(N) = \operatorname{rank}_S(N)$ is attained.

As we saw in Example 4.4.4, a matrix factorization of the form $N^\flat$, obtained by computing multiplication by z on an MCM $R^\sharp$-module N, can be non-reduced. We will show below that the matrix representing multiplication by z on N contains unit entries precisely when $\mu_{R^\sharp}(N) < \operatorname{rank}_S(N)$. In other words, the restriction of the functor $(-)^\flat : \mathrm{MCM}(R^\sharp) \to \mathrm{MF}_S^d(f)$ to the subcategory of MCM $R^\sharp$-modules satisfying $\mu_{R^\sharp}(N) = \operatorname{rank}_S(N)$ produces only reduced matrix factorizations of f with d factors. Conversely, the image of the functor $(-)^\sharp : \mathrm{MF}_S^d(f) \to \mathrm{MCM}(R^\sharp)$, restricted to the subcategory of reduced matrix factorizations of f, consists exactly of the MCM $R^\sharp$-modules N satisfying $\mu_{R^\sharp}(N) = \operatorname{rank}_S(N)$.

Lemma 4.6.6. *Let N be an MCM $R^\sharp$-module and assume that $f + z^d$ is irreducible. Then N is a finitely generated free S-module satisfying*

$$\mu_{R^\sharp}(N) \leq \operatorname{rank}_S(N) = d \cdot \operatorname{rank}_{R^\sharp}(N) = \operatorname{rank}_S(R^\sharp) \cdot \operatorname{rank}_{R^\sharp}(N).$$

Proof. Let $(\Phi : S[\![z]\!]^n \to S[\![z]\!]^n, \Psi : S[\![z]\!]^n \to S[\![z]\!]^n) \in \mathrm{MF}^2_{S[\![z]\!]}(f + z^d)$ be a matrix factorization of $f + z^d$ such that Φ gives a minimal presentation of $\operatorname{cok} \Phi = N$. Since Φ is minimal, $n = \mu_{R^\sharp}(N)$. Then $\det \Phi = u(f + z^d)^k$ for some $1 \leq k \leq n$ and some unit $u \in S[\![z]\!]$. Recall that $k = \operatorname{rank}_{R^\sharp}(N)$ by [Eis80, Propoistion 5.6]. By tensoring with $S = S[\![z]\!]/(z)$, we find that $\det \bar{\Phi} = v \cdot f^k$, where $\bar{\Phi} = \Phi \otimes_{S[\![z]\!]} 1_S$ and $v \in S$ is a unit. Moreover, $\bar{\Phi}$ is injective, since $\bar{\Phi}\bar{\Psi} = f \cdot 1_{S^n} = \bar{\Psi}\bar{\Phi}$, and we have a minimal presentation of N/zN over S:

$$0 \longrightarrow S^n \xrightarrow{\bar{\Phi}} S^n \longrightarrow N/zN \longrightarrow 0.$$

On the other hand, since N is MCM over $R^\sharp$, it is finitely generated and free as an S-

module. Let $r = \operatorname{rank}_S(N)$ and consider the map $\varphi : S^r \to S^r$ representing multiplication by z on N. This map also gives a presentation of N/zN over S, though the presentation may not be minimal as we saw in Example 4.4.4. Thus, there exists a commutative diagram with vertical isomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^r & \xrightarrow{\begin{pmatrix} \bar{\Phi} & 0 \\ 0 & I_{r-n} \end{pmatrix}} & S^r & \longrightarrow & N/zN & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & S^r & \xrightarrow{\ \varphi\ } & S^r & \longrightarrow & N/zN & \longrightarrow & 0. \end{array} \tag{4.6.2}$$

This implies that $\mu_R(N/zN) \leq r$, where $R = S/(f)$ as usual. The inequality now follows from the fact that $\mu_R(N/zN) = \mu_{R^\sharp}(N)$. Furthermore, the diagram implies that $\det \varphi = v' \cdot f^k$ for some unit v'. However, since $\varphi^d = -f \cdot I_r$, we have that, up to units, $f^r = (\det \varphi)^d = f^{kd}$. Thus, $\operatorname{rank}_S(N) = r = dk = d \cdot \operatorname{rank}_{R^\sharp}(N)$. □

Lemma 4.6.7. *Assume $f + z^d$ is irreducible. Let N be an MCM $R^\sharp$-module and let $X \in \operatorname{MF}_S^d(f)$. Then $\mu_{R^\sharp}(N) = \operatorname{rank}_S(N)$ if and only if $N^\flat \in \operatorname{MF}_S^d(f)$ is reduced, and $X^\sharp$ satisfies $\mu_{R^\sharp}(X^\sharp) = \operatorname{rank}_S(X^\sharp)$ if and only if X is reduced.*

Proof. Let $N \in \operatorname{MCM}(R^\sharp)$ and set $r = \operatorname{rank}_S(N)$. Let $\varphi : S^r \to S^r$ be the S-linear map representing multiplication by z on N. Then the presentation of N/zN given by φ in (4.6.2) is minimal if and only if $r = \operatorname{rank}_S(N) = \mu_R(N/zN)$. Since $\mu_R(N/zN) = \mu_{R^\sharp}(N)$, we have that φ is minimal if and only if $\operatorname{rank}_S(N) = \mu_{R^\sharp}(N)$. This proves the first statement since $N^\flat = (\mu\varphi, \mu\varphi, \cdots, \mu\varphi) \in \operatorname{MF}_S^d(f)$.

By Proposition 4.3.5, $X^{\sharp\flat} \cong \bigoplus_{k \in \mathbb{Z}_d} T^k X$ which is reduced if and only if X is reduced. The second statement now follows from the first by taking $N = X^\sharp \in \operatorname{MCM}(R^\sharp)$. □

Lemma 4.6.7 gives us a specialization of Corollary 4.3.6 and Theorem 4.3.7.

Proposition 4.6.8. *Assume $f + z^d$ is irreducible.*

(i) For any reduced $X \in \mathrm{MF}_S^d(f)$*, there exists* $N \in \mathrm{MCM}(R^\sharp)$ *satisfying* $\mathrm{rank}_S(N) = \mu_{R^\sharp}(N)$ *such that* X *is isomorphic to a direct summand of* $N^\flat$*.*

(ii) For any $N \in \mathrm{MCM}(R^\sharp)$ *satisfying* $\mathrm{rank}_S(N) = \mu_{R^\sharp}(N)$*, there exists reduced* $X \in \mathrm{MF}_S^d(f)$ *such that* N *is isomorphic to a direct summand of* $X^\sharp$*.*

In particular, f *has finite reduced* d*-MF type if and only if there are, up to isomorphism, only finitely many indecomposable MCM* $R^\sharp$*-modules* N *satisfying* $\mathrm{rank}_S(N) = \mu_{R^\sharp}(N)$*.*

Proof. Both (i) and (ii) follow from Lemma 4.6.7 and Proposition 4.3.5. The final statement follows as in the proof of Theorem 4.3.7 by noticing that a matrix factorization $Y \in \mathrm{MF}_S^d(f)$ is reduced if and only if every summand of Y is reduced and that an MCM $R^\sharp$-module N satisfies $\mu_{R^\sharp}(N) = \mathrm{rank}_S(N)$ if and only if every summand of N satisfies the same equality. □

For a module M over a local ring A, we let $e(M)$ denote the multiplicity of M. If M is an MCM A-module, there is a well known inequality $\mu_A(M) \le e(M)$. The class of MCM modules satisfying $\mu_A(M) = e(M)$ are called *Ulrich modules*. For background on Ulrich modules we refer the reader to [Bea18], [BHU87], [HK87], and [HUB91]. If A is a domain, then we may compute the multiplicity of M as $e(M) = e(A) \cdot \mathrm{rank}_A(M)$.

In the case of the d-fold branched cover of R, we have the following connection between reduced d-fold matrix factorizations of f and Ulrich modules over $R^\sharp$. We let $\mathrm{ord}(f)$ denote the maximal integer e such that $f \in \mathfrak{n}^e$.

Corollary 4.6.9. *Assume* $d \le \mathrm{ord}(f)$ *and that* $f + z^d$ *is irreducible. Let* $N \in \mathrm{MCM}(R^\sharp)$*. Then* N *is an Ulrich* $R^\sharp$*-module if and only if* $N^\flat \in \mathrm{MF}_S^d(f)$ *is a reduced matrix factorization of* f*.*

In particular, f *has finite reduced* d*-MF type if and only if there are, up to isomorphism, only finitely many indecomposable Ulrich* $R^\sharp$*-modules.*

Proof. Since $d \le \mathrm{ord}(f)$, the multiplicity of $R^\sharp = S[\![z]\!]/(f + z^d)$ is d. Hence, an MCM $R^\sharp$-module N is Ulrich if and only if $\mu_{R^\sharp}(N) = d \cdot \mathrm{rank}_{R^\sharp}(N)$. By Lemma 4.6.6, the quantity

$d \cdot \operatorname{rank}_{R^\sharp}(N)$ is equal to the rank of N as a free S-module. Thus, N is Ulrich if and only if $\mu_{R^\sharp}(N) = \operatorname{rank}_S(N)$. Both statements now follow from Proposition 4.6.7. □

Remark 4.6.10. In the case $d = 2$, the condition $\operatorname{rank}_S(N) = \mu_{R^\sharp}(N)$ is redundant. An MCM $R^\sharp = S[\![z]\!]/(f + z^2)$-module N satisfies $\operatorname{rank}_S(N) = \mu_{R^\sharp}(N)$ if and only if N has no summands isomorphic to $R^\sharp$ (this follows from the proof of [LW12, Lemma 8.17 *(iii)*]). In other words, the conclusion of Proposition 4.6.8 in the case $d = 2$ is simply a restatement of Theorem 4.3.2. Furthermore, Corollary 4.6.9 implies that any MCM $R^\sharp$-module with no free summands is an Ulrich module. Since, in the case $d = 2$, the multiplicity of $R^\sharp$ is 2 this is a known result of Herzog-Kühl [HK87, Corollary 1.4].

Example 4.6.11. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 and consider the one-dimensional hypersurface ring

$$R_{a,i} = \mathbf{k}[\![x, y]\!]/(x^a + y^{a+i}), \quad a \geq 2, i \geq 0.$$

If $i = 1$ or $i = 2$, then, by [HUB91, Theorem A.3], $R_{a,i}$ has only finitely many isomorphism classes of indecomposable Ulrich modules. Set $S = \mathbf{k}[\![y]\!]$ and consider $R_{a,i}$ as the a-fold branched cover of $R = \mathbf{k}[\![y]\!]/(y^{a+i})$. Since $e(R_{a,i}) = a$, Corollary 4.6.9 implies that y^{a+i}, for $i \in \{1, 2\}$, has only finitely many isomorphism classes of reduced indecomposable a-fold matrix factorizations. In other words, y^{a+i} has finite reduced a-MF type for $i = 1, 2$ and any $a \geq 2$.

The methods in [HUB91, Theorem A.3] can be used to compute the isomorphism classes of indecomposable reduced matrix factorizations of y^{a+i}. For instance, let $a \geq 2$ and $i = 1$. Then $R_{a,1} \cong \mathbf{k}[\![t^a, t^{a+1}]\!]$ and t^a is a minimal reduction of the maximal ideal $\mathfrak{m}$ of $R_{a,1}$. Hence, $R'_{a,1} = R_{a,1}[\{\frac{r}{t^a} : r \in \mathfrak{m}\}] = \mathbf{k}[\![t]\!]$ is the first quadratic transform of $R_{a,1}$. By [HUB91, Corollary A.1], an $R_{a,1}$-module M is Ulrich over $R_{a,1}$ if and only if it is MCM over $R'_{a,1}$. Since $R'_{a,1} = \mathbf{k}[\![t]\!]$ is a regular local ring, the only indecomposable MCM $R'_{a,1}$-module is $R'_{a,1}$ itself.

As an $S \cong \mathbf{k}[\![t^a]\!]$-module, $R'_{a,1} = \mathbf{k}[\![t]\!]$ is free with basis given by $\{1, t, t^2, \ldots, t^{a-1}\}$. Thus, multiplication by $x = t^{a+1}$ on the basis $\{1, t, \ldots, t^{a-1}\}$ is given by the mapping

$$t^k \mapsto t^{a+1+k} = t^a t^{k+1}$$

for $0 \leq k \leq a-1$. Since $y = t^a$, it follows that multiplication by x on the MCM $R_{a,1}$-module $R'_{a,1}$ is given by the $a \times a$ matrix with entries in $\mathbf{k}[\![y]\!]$

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & y^2 \\ y & 0 & \cdots & 0 & 0 \\ 0 & y & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & y & 0 \end{pmatrix}.$$

It follows that $(R'_{a,1})^\flat \cong \bigoplus_{i \in \mathbb{Z}_a} T^i((y^2, y, y, \ldots, y)) \in \mathrm{MF}^a_{\mathbf{k}[\![y]\!]}(y^{a+1})$. By Proposition 4.6.8 and Corollary 4.6.9, the matrix factorization $(y^2, y, y, \ldots, y) \in \mathrm{MF}^a_{\mathbf{k}[\![y]\!]}(y^{a+1})$, and its corresponding shifts, are the only indecomposable reduced matrix factorizations of y^{a+1} with a factors.

Notice that for $a \geq 4$, the polynomial y^{a+1} does not appear on the list given in Theorem 4.4.2 for any $d > 2$. Thus, the conclusions of this example imply that y^{a+1} has infinitely many isomorphism classes of indecomposable matrix factorizations with a factors but only finitely many which are reduced.

The last example shows the necessity of the assumption $d \leq \mathrm{ord}(f)$ in Corollary 4.6.9.

Example 4.6.12. Let $\mathbf{k}$ be algebraically closed of characteristic 0. Set $S = \mathbf{k}[\![x]\!]$, $f = x^3$, and $R = S/(f)$. The hypersurface ring $R^\sharp = \mathbf{k}[\![x, y]\!]/(x^3 + y^4)$ is the same ring given in Example 4.4.4, however, here we are viewing $R^\sharp$ as the 4-fold branched cover of $R = \mathbf{k}[\![x]\!]/(x^3)$. Again using the notation of [Yos90, p. 9.13], we take $B = \operatorname{cok}\beta$ where

$$\beta = \begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{pmatrix}.$$

The MCM $R^\sharp$-module B is, in this case, free of rank 4 over $S = \mathbf{k}[\![x]\!]$. In particular, if e_1, e_2, and e_3 are the images of the standard basis on $S[\![y]\!]^3$, then an S-basis for B is $\{e_1, e_2, e_3, ye_2\}$. Multiplication by y on B is given by the S-matrix

$$\varphi = \begin{pmatrix} 0 & 0 & x & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Notice that B is an Ulrich $R^\sharp$-module but multiplication by y on B is given by a non-reduced matrix. In other words, the condition $d \leq \operatorname{ord}(f)$ in Proposition 4.6.8 is necessary.

5 | Morphism Categories of MCM modules

Let S be a regular local ring, $0 \neq f$ a non-invertible element of S, and set $R = S/(f)$. The main goal of this chapter is to generalize Eisenbud's correspondence (Theorem 1.2.7), between matrix factorizations of f with 2 factors and maximal Cohen-Macaulay R-modules, to the case of matrix factorizations of f with $d > 2$ factors.

5.1 Morphism categories

Definition 5.1.1. Let A be a local ring and $M, N \in \mathrm{MCM}(A)$. Fix $n \geq 1$.

(i) An A-homomorphism $g : M \to N$ is called *admissible* if both $\operatorname{cok} g$ and $\ker g$ are in $\mathrm{MCM}(A)$. Let $\mathrm{Mor}_n(A)$ denote the category of sequences of admissible homomorphisms of length n. In other words, an object $\xi \in \mathrm{Mor}_n(A)$ is a sequence

$$\xi = \big(M_n \xrightarrow{g_{n-1}} M_{n-1} \xrightarrow{g_{n-2}} \cdots \xrightarrow{g_2} M_2 \xrightarrow{g_1} M_1\big) \tag{5.1.1}$$

such that $M_i \in \mathrm{MCM}(A)$ for $i = 1, 2, \ldots, n$ and g_i is an admissible homomorphism for $i = 1, 2, \ldots, n-1$. A morphism $\beta = (\beta_1, \beta_2, \ldots, \beta_n) : \xi \to \xi'$ between objects $\xi, \xi' \in \mathrm{Mor}_n(A)$ is a commutative diagram:

$$\begin{array}{ccccccccc}
M_n & \xrightarrow{g_{n-1}} & M_{n-1} & \xrightarrow{g_{n-2}} & \cdots & \xrightarrow{g_2} & M_2 & \xrightarrow{g_1} & M_1 \\
\downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n-1}} & & & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} \\
M'_n & \xrightarrow{g'_{n-1}} & M'_{n-1} & \xrightarrow{g'_{n-2}} & \cdots & \xrightarrow{g'_2} & M'_2 & \xrightarrow{g'_1} & M'_1.
\end{array}$$

(ii) An A-homomorphism $g : M \to N$ is called an *admissible monomorphism* if it is admissible and injective. The *monomorphism category* of A, denoted $\mathcal{S}_n(A)$, is the full subcategory of $\mathrm{Mor}_n(A)$ consisting of sequences of admissible monomorphisms of length n.

(iii) Dually, $g : M \to N$ is called an *admissible epimorphism* if it is admissible and surjective. The *epimorphism category* of A, denoted $\mathcal{F}_n(A)$, is the full subcategory of $\mathrm{Mor}_n(A)$ consisting of sequences of admissible epimorphisms of length n.

The Depth Lemma [LW12, Lemma A.4] ensures that the subcategories $\mathcal{F}_n(A)$ and $\mathcal{S}_n(A)$ are well behaved. In particular, $\mathrm{MCM}(A)$ is closed under extensions and an admissible epimorphism in $\mathrm{MCM}(A)$ is just an epimorphism of A-modules. For this reason, and the following lemma, we will focus our attention on the category $\mathcal{F}_n(A)$.

Lemma 5.1.2. *For any $n \geq 1$, there is an equivalence of categories $\mathcal{F}_n(A) \approx \mathcal{S}_n(A)$.*

Proof. This follows directly from the fact that $\mathrm{MCM}(A)$ is closed under extensions. In particular, the equivalence is given by the functor which sends $\xi \in \mathcal{F}_n(A)$, of the form (5.1.1), to the sequence

$$\ker g_{n-1} \hookrightarrow \ker(g_{n-2}g_{n-1}) \hookrightarrow \cdots \hookrightarrow \ker(g_1 g_2 \cdots g_{n-1}) \hookrightarrow M_n$$

in $\mathcal{S}_n(A)$. □

In the case of an Artin algebra Λ, the analogous morphism categories of Λ-modules have been studied thoroughly. Ringel and Schmidmeier considered the case of the "submodule category" ($n = 2$) in the series of papers [RS06], [RS07], and [RS08]. The general case ($n > 2$) has also received a great deal of attention. We refer the reader to [Sim07], [Che11], [RZ14], and [XZZ14] for more information.

5.2 The epimorphism category of a hypersurface ring

Let S be a regular local ring, $0 \neq f \in S$ a non-unit, and fix an integer $d \geq 2$. For the rest of this chapter, we will consider the morphism categories defined in (5.1.1) in the case of the hypersurface ring $R = S/(f)$. In light of Lemma 5.1.2, we need only investigate one of $\mathcal{S}_n(R)$ or $\mathcal{F}_n(R)$. We have chosen to present the main result of this chapter from the perspective of the epimorphism category.

Recall the indecomposable projective matrix factorizations $\mathcal{P}_i, i \in \mathbb{Z}_d$, from Chapter 2.1. Namely, $\mathcal{P}_i = (1, 1, ..., f, ..., 1, 1)$ where the i-th component is multiplication by f on S while the rest are the identity on S. Given an additive category $\mathcal{C}$ and a set of objects $\mathcal{B}$ in $\mathcal{C}$, we let $\mathcal{C}/\mathcal{B}$ denote the category which has the same objects as $\mathcal{C}$, and has morphisms which factor through direct sums of objects in $\mathcal{B}$ identified with zero. Before stating the main result of this chapter, we recall a more precise version of Eisenbud's Theorem (1.2.7).

Theorem 5.2.1 ([Eis80], Theorem 7.4 [Yos90])**.** *The functor* $\operatorname{cok} : \mathrm{MF}^2_S(f) \to \mathrm{MCM}(R)$, *given by* $(\varphi, \psi) \in \mathrm{MF}^2_S(f) \longmapsto \operatorname{cok}\varphi \in \mathrm{MCM}(R)$, *induces an equivalence of categories:*

$$\mathrm{MF}^2_S(f)/\{\mathcal{P}_2\} \approx \mathrm{MCM}(R).$$

Furthermore, this induces an equivalence between stable categories:

$$\underline{\mathrm{MF}}^2_S(f) = \mathrm{MF}^2_S(f)/\{\mathcal{P}_1, \mathcal{P}_2\} \approx \mathrm{MCM}(R)/\{R\}.$$

The rest of this chapter is dedicated to proving our main result:

Theorem 5.2.2. *There is an equivalence of categories* $\mathrm{MF}^d_S(f)/\{\mathcal{P}_d\} \approx \mathcal{F}_{d-1}(R)$.

Remark 5.2.3. The cases $d = 2$ and $d = 3$ of Theorem 5.2.2 are known.

(i) If $d = 2$, then $\mathcal{F}_1(R) = \mathrm{MCM}(R)$ is the category of MCM R-modules. In this case, Theorem 5.2.2 coincides with Eisennbud's fundamental theorem on matrix factorizations

(Theorem 5.2.1 above).

(ii) If $d = 3$, then $\mathcal{F}_2(R)$ can be identified with the category of short exact sequences of MCM R-modules. In this case, Hopkins [Hop21, Theorem 3.1.4] proved that the category of 3-fold matrix factorizations of f, modulo the projective factorization $\mathcal{P}_3 = (1, 1, f)$ in $\mathrm{MF}^3_S(f)$, is equivalent to the category $\mathcal{F}_2(R)$.

The key step in the proof of Hopkins' result is the construction of a 3-fold matrix factorization from a short exact sequence of MCM R-modules using the horseshoe lemma. The following proposition is well-known (see [BL07, Proposition 3.5] or [LW12, Remark 8.9]) and gives an equivalent description of the factorization constructed by Hopkins.

Proposition 5.2.4. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of MCM R-modules. For any matrix factorizations (φ', ψ') and (φ'', ψ'') corresponding to M' and M'' respectively, there exists a morphism $(\alpha, \beta) : (\psi'', \varphi'') \to (\varphi', \psi')$ such that*

$$\operatorname{cok}\left(\begin{pmatrix} \varphi' & \alpha \\ 0 & \varphi'' \end{pmatrix}, \begin{pmatrix} \psi' & -\beta \\ 0 & \psi'' \end{pmatrix}\right) \cong M.$$

In particular, this matrix factorization can be factored further into

$$\left(\begin{pmatrix} I_n & 0 \\ 0 & \varphi'' \end{pmatrix}, \begin{pmatrix} \varphi' & \alpha \\ 0 & I_m \end{pmatrix}, \begin{pmatrix} \psi' & -\beta \\ 0 & \psi'' \end{pmatrix}\right) \in \mathrm{MF}^3_S(f)$$

where I_n and I_m are identity matrices of the appropriate sizes. □

5.2.1 The proof of the main result

Let $d \geq 2$. Given a matrix factorization $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}^d_S(f)$, define $X_j = \operatorname{cok}(\varphi_1\varphi_2 \cdots \varphi_{j-1}\varphi_j) \in \mathrm{MCM}(R)$ for each $j \in \mathbb{Z}_d$. Additionally, let π_j denote the canonical projection map $\pi_j : F_1 \to X_j$ given by the short exact sequence:

$$0 \longrightarrow F_{j+1} \xrightarrow{\varphi_1\varphi_2\cdots\varphi_j} F_1 \xrightarrow{\pi_j} X_j \longrightarrow 0.$$

Define a functor $\Psi_d : \mathrm{MF}_S^d(f) \to \mathcal{F}_{d-1}(R)$ which sends a matrix factorization $X \in \mathrm{MF}_S^d(f)$ to the sequence of surjections

$$X_{d-1} \xrightarrow{\rho_{d-2}} X_{d-2} \xrightarrow{\rho_{d-3}} \cdots \xrightarrow{\rho_2} X_2 \xrightarrow{\rho_1} X_1$$

where ρ_j is the unique map completing the diagram

$$\begin{CD}
0 @>>> F_{j+2} @>{\varphi_1\varphi_2\cdots\varphi_{j+1}}>> F_1 @>{\pi_{j+1}}>> X_{j+1} @>>> 0\\
@. @VV{\varphi_{j+1}}V @| @VV{\rho_j}V @.\\
0 @>>> F_{j+1} @>{\varphi_1\varphi_2\cdots\varphi_j}>> F_1 @>{\pi_j}>> X_j @>>> 0.
\end{CD}$$

Given a morphism $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) : X \to X'$, the morphism $\Psi_d(\alpha)$ is given by

$$\begin{CD}
X_{d-1} @>{\rho_{d-2}}>> X_{d-2} @>{\rho_{d-3}}>> \cdots @>{\rho_2}>> X_2 @>{\rho_1}>> X_1\\
@VV{(\alpha_1,\alpha_d)}V @VV{(\alpha_1,\alpha_{d-1})}V @. @VV{(\alpha_1,\alpha_3)}V @VV{(\alpha_1,\alpha_2)}V\\
X'_{d-1} @>{\rho'_{d-2}}>> X'_{d-2} @>{\rho'_{d-3}}>> \cdots @>{\rho'_2}>> X'_2 @>{\rho'_1}>> X'_1
\end{CD}$$

where (α_1, α_j) is the unique map induced by the diagram

$$\begin{CD}
0 @>>> F_j @>{\varphi_1\varphi_2\cdots\varphi_{j-1}}>> F_1 @>{\pi_j}>> X_j @>>> 0\\
@. @VV{\alpha_j}V @VV{\alpha_1}V @VV{(\alpha_1,\alpha_j)}V @.\\
0 @>>> F'_j @>{\varphi'_1\varphi'_2\cdots\varphi'_{j-1}}>> F'_1 @>{\pi'_j}>> X'_j @>>> 0.
\end{CD}$$

In order to prove Theorem 5.2.2, we show that Ψ_d is fully faithful and dense. We start with a lemma which will be used to show the density of Ψ_d.

Lemma 5.2.5. *Let $X = (\varphi_1, \ldots, \varphi_{d-1}) \in \mathrm{MF}_S^{d-1}(f)$ and assume there exists an MCM R-module M with a surjection $g : M \to X_{d-2}$. Let $\xi \in \mathcal{F}_{d-1}(R)$ denote the resulting sequence* *of length $d-1$:*

$$\xi = \big(M \xrightarrow{g} X_{d-2} \xrightarrow{\rho_{d-3}} X_{d-3} \xrightarrow{\rho_{d-4}} \cdots \xrightarrow{\rho_2} X_2 \xrightarrow{\rho_1} X_1\big).$$

Then there exists $Y \in \mathrm{MF}_S^d(f)$ such that $\Psi_d(Y) \cong \xi$.

Proof. For simplicity, set $\Phi_j = \varphi_1\varphi_2\cdots\varphi_{j-1}\varphi_j$ for each $1 \le j \le d-2$. Let $K = \ker g$ and pick a matrix factorization $(\varphi_K : G_K \to F_K, \psi_K : F_K \to G_K) \in \mathrm{MF}_S^2(f)$ such that $\operatorname{cok}\varphi_K \cong K$. By Proposition 5.2.4, there exists a morphism of 2-fold matrix factorizations $(\alpha, \beta) : (\varphi_{d-1}, \Phi_{d-2}) \to (\varphi_K, \psi_K)$ such that

$$M \cong \operatorname{cok}\left(\begin{pmatrix} \varphi_K & \alpha \\ 0 & \Phi_{d-2} \end{pmatrix}, \begin{pmatrix} \psi_K & -\beta \\ 0 & \varphi_{d-1} \end{pmatrix}\right) \tag{5.2.1}$$

and a commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_K & \xrightarrow{u'} & G_K \oplus F_{d-1} & \xrightarrow{v'} & F_{d-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_K} & & \downarrow{\scriptstyle \begin{pmatrix} \varphi_K & \alpha \\ 0 & \Phi_{d-2} \end{pmatrix}} & & \downarrow{\scriptstyle \Phi_{d-2}} & & \\
0 & \longrightarrow & F_K & \xrightarrow{u} & F_K \oplus F_1 & \xrightarrow{v} & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{d-2}} & & \\
0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{g} & X_{d-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.2.2}$$

where u, u' are the canonical inclusions and v, v' are the canonical projections. The matrix factorization (5.2.1) gives rise to a d-fold factorization $Y \in \mathrm{MF}_S^d(f)$ given by

$$Y = \left(\begin{pmatrix} 1_{F_K} & 0 \\ 0 & \varphi_1 \end{pmatrix}, \begin{pmatrix} 1_{F_K} & 0 \\ 0 & \varphi_2 \end{pmatrix}, \ldots, \begin{pmatrix} 1_{F_K} & 0 \\ 0 & \varphi_{d-2} \end{pmatrix}, \begin{pmatrix} \varphi_K & \alpha \\ 0 & 1_{F_{d-1}} \end{pmatrix}, \begin{pmatrix} \psi_K & -\beta \\ 0 & \varphi_{d-1} \end{pmatrix} \right).$$

What is left to show is that $\Psi_d(Y) \cong \xi$. From the commutative diagram (5.2.2), we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_K \oplus F_{d-1} & \xrightarrow{\begin{pmatrix} \varphi_K & \alpha \\ 0 & \Phi_{d-2} \end{pmatrix}} & F_K \oplus F_1 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \begin{pmatrix} \varphi_K & \alpha \\ 0 & 1_{F_{d-1}} \end{pmatrix}} & & \Big\| & & \Big\downarrow {\scriptstyle g} & & \\ 0 & \longrightarrow & F_K \oplus F_{d-1} & \xrightarrow[\begin{pmatrix} 1_{F_K} & 0 \\ 0 & \Phi_{d-2} \end{pmatrix}]{} & F_K \oplus F_1 & \xrightarrow{\pi_{d-2} v} & X_{d-2} & \longrightarrow & 0 \end{array} \tag{5.2.3}$$

which has exact rows and also commutes. As above, let $\rho'_j : Y_{j+1} \to Y_j$, $1 \le j \le d-2$, denote the canonical map induced by the matrix factorization $Y \in \mathrm{MF}_S^d(f)$. Given a homomorphism $h : N \to N'$, let $\bar{h}$ denote the induced map $\bar{h} : N/\ker h \to N'$. The diagram (5.2.3) induces a commutative diagram

$$\begin{array}{ccc} Y_{d-1} & \xrightarrow{\bar{\pi}} & M \\ \Big\downarrow {\scriptstyle \rho'_{d-2}} & & \Big\downarrow {\scriptstyle g} \\ Y_{d-2} & \xrightarrow{\overline{\pi_{d-2} v}} & X_{d-2} \end{array}$$

with horizontal isomorphisms.

Similarly, for each $1 \le j \le d-3$, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_K \oplus F_{j+1} & \xrightarrow{\begin{pmatrix} 1_{F_K} & 0 \\ 0 & \Phi_j \end{pmatrix}} & F_K \oplus F_1 & \xrightarrow{\pi_j v} & X_j & \longrightarrow & 0 \\ & & \Big\downarrow \begin{pmatrix} 1_{F_K} & 0 \\ 0 & \varphi_j \end{pmatrix} & & \Big\| & & \Big\downarrow \rho_{j-1} & & \\ 0 & \longrightarrow & F_K \oplus F_j & \xrightarrow[\begin{pmatrix} 1_{F_K} & 0 \\ 0 & \Phi_{j-1} \end{pmatrix}]{} & F_K \oplus F_1 & \xrightarrow{\pi_{j-1} v} & X_{j-1} & \longrightarrow & 0 \end{array}$$

induces isomorphisms $\overline{\pi_j v} : Y_j \to X_j$ and $\overline{\pi_{j-1} v} : Y_{j-1} \to X_{j-1}$ such that $\rho_{j-1}\overline{\pi_j v} = \overline{\pi_{j-1} v}\rho'_{j-1}$. Thus, we have an isomorphism in $\mathcal{F}_{d-1}(R)$:

$$\begin{array}{ccccccccccc} Y_{d-1} & \xrightarrow{\rho'_{d-2}} & Y_{d-2} & \xrightarrow{\rho'_{d-3}} & Y_{d-3} & \xrightarrow{\rho'_{d-4}} & \cdots & \xrightarrow{\rho'_2} & Y_2 & \xrightarrow{\rho'_1} & Y_1 \\ \Big\downarrow \overline{\pi} & & \Big\downarrow \overline{\pi_{d-2} v} & & \Big\downarrow \overline{\pi_{d-3} v} & & & & \Big\downarrow \overline{\pi_2 v} & & \Big\downarrow \overline{\pi_1 v} \\ M & \xrightarrow{g} & X_{d-2} & \xrightarrow{\rho_{d-3}} & X_{d-3} & \xrightarrow{\rho_{d-3}} & \cdots & \xrightarrow{\rho_2} & X_2 & \xrightarrow{\rho_1} & X_1. \end{array}$$

□

Proposition 5.2.6. *The functor* $\Psi_d : \mathrm{MF}_S^d(f) \to \mathcal{F}_{d-1}(R)$ *is dense.*

Proof. We prove that Ψ_d is dense by induction on $d \ge 2$. As we mentioned in Remark 5.2.3, the cases $d = 2$ and $d = 3$ hold. Assume $d > 3$ and that Ψ_{d-1} is dense. Let $\xi \in \mathcal{F}_{d-1}(R)$, that is,

$$\xi = M_{d-1} \xrightarrow{g_{d-2}} M_{d-2} \xrightarrow{g_{d-3}} \cdots \xrightarrow{g_2} M_2 \xrightarrow{g_1} M_1$$

for MCM R-modules $M_1, \ldots, M_{d-1}$ and surjective homomorphisms $g_1, \ldots, g_{d-2}$. By induction, there exists a matrix factorization $X \in \mathrm{MF}_S^{d-1}(f)$ such that $\Psi_{d-1}(X)$ is isomorphic to the sequence of surjections of length $d-2$ starting at M_{d-2}. In other words, there exists isomorphisms $\gamma_1, \ldots, \gamma_{d-2}$ and a commutative diagram

$$\begin{array}{ccccccccc}
M_{d-2} & \xrightarrow{g_{d-3}} & M_{d-3} & \xrightarrow{g_{d-4}} & \cdots & \xrightarrow{g_2} & M_2 & \xrightarrow{g_1} & M_1 \\
\downarrow{\scriptstyle \gamma_{d-2}} & & \downarrow{\scriptstyle \gamma_{d-3}} & & & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_1} \\
X_{d-2} & \xrightarrow{\rho_{d-3}} & X_{d-3} & \xrightarrow{\rho_{d-4}} & \cdots & \xrightarrow{\rho_2} & X_2 & \xrightarrow{\rho_1} & X_1.
\end{array}$$

This isomorphism in $\mathcal{F}_{d-2}(R)$ extends to an isomorphism in $\mathcal{F}_{d-1}(R)$:

$$\begin{array}{ccccccccccc}
\xi = M_{d-1} & \xrightarrow{g_{d-2}} & M_{d-2} & \xrightarrow{g_{d-3}} & M_{d-3} & \xrightarrow{g_{d-4}} & \cdots & \xrightarrow{g_2} & M_2 & \xrightarrow{g_1} & M_1 \\
\| & & \downarrow{\scriptstyle \gamma_{d-2}} & & \downarrow{\scriptstyle \gamma_{d-3}} & & & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_1} \\
\xi' = M_{d-1} & \xrightarrow{\gamma_{d-2} g_{d-2}} & X_{d-2} & \xrightarrow{\rho_{d-3}} & X_{d-3} & \xrightarrow{\rho_{d-4}} & \cdots & \xrightarrow{\rho_2} & X_2 & \xrightarrow{\rho_1} & X_1.
\end{array}$$

Now, we may apply Lemma 5.2.5 to obtain a d-fold matrix factorization $Y \in \mathrm{MF}_S^d(f)$ such that $\Psi_d(Y) \cong \xi'$ which is in turn isomorphic to ξ. □

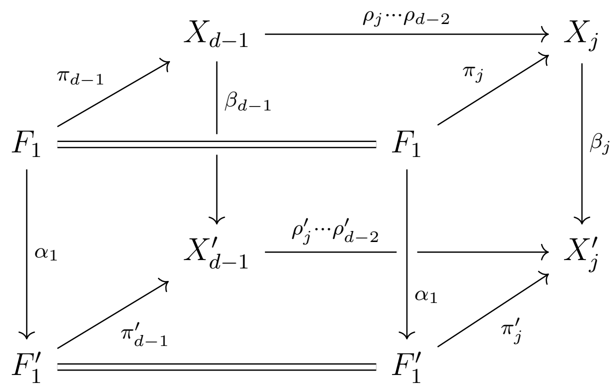
Proposition 5.2.7. *The induced functor* $\Psi_d : \mathrm{MF}_S^d(f)/\{\mathcal{P}_d\} \to \mathcal{F}_{d-1}(R)$ *is fully faithful.*

Proof. Since $\Psi_d(\mathcal{P}_d) = 0$, there is an induced functor $\mathrm{MF}_S^d(f)/\{\mathcal{P}_d\} \to \mathcal{F}_{d-1}(R)$, which we will also call Ψ_d. In order to show that Ψ_d is full, let $X, X' \in \mathrm{MF}_S^d(f)$ and assume $\beta = (\beta_1, \beta_2, \ldots, \beta_{d-1}) : \Psi_d(X) \to \Psi_d(X')$ is a morphism in $\mathcal{F}_{d-1}(R)$. Since F_1 is a free S-module, there exist α_1 and α_d making the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_d & \xrightarrow{\varphi_1\varphi_2\cdots\varphi_{d-1}} & F_1 & \xrightarrow{\pi_{d-1}} & X_{d-1} & \longrightarrow & 0 \\
 & & \vdots{\scriptstyle \alpha_d} & & \vdots{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_{d-1}} & & \\
0 & \longrightarrow & F'_d & \xrightarrow{\varphi'_1\varphi'_2\cdots\varphi'_{d-1}} & F'_1 & \xrightarrow{\pi'_{d-1}} & X'_{d-1} & \longrightarrow & 0
\end{array}$$

commute.

Next, we claim that $\beta_j\pi_j = \pi'_j\alpha_1$ for each $1 \le j \le d-2$. To see this, consider the diagram



Since $\beta_{d-1}\pi_{d-1} = \pi'_{d-1}\alpha_1$ we can see that each face of this cube commutes except possibly the right most side face. However, using the commutativity of the other faces, we have that

$$\beta_j\pi_j = \beta_j\rho_j\cdots\rho_{d-2}\pi_{d-1} = \rho'_j\cdots\rho'_{d-2}\beta_{d-1}\pi_{d-1} = \rho'_j\cdots\rho'_{d-2}\pi'_{d-1}\alpha_1 = \pi'_j\alpha_1.$$

It follows that, for each $1 \le j \le d-1$, there exists a homomorphism $\alpha_{j+1} : F_{j+1} \to F'_{j+1}$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{j+1} & \xrightarrow{\varphi_1\varphi_2\cdots\varphi_j} & F_1 & \xrightarrow{\pi_j} & X_j & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{j+1}} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_j} & & \\ 0 & \longrightarrow & F'_{j+1} & \xrightarrow{\varphi'_1\varphi'_2\cdots\varphi'_j} & F'_1 & \xrightarrow{\pi'_j} & X'_j & \longrightarrow & 0 \end{array}$$

commutes.

Finally, we show that $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ forms a morphism of matrix factorizations $X \to X'$. To see this, let $1 \le k \le d-1$ and notice that

$$\varphi'_1\varphi'_2\cdots\varphi'_{k-1}\varphi'_k\alpha_{k+1} = \alpha_1\varphi_1\varphi_2\cdots\varphi_{k-1}\varphi_k = \varphi'_1\varphi'_2\cdots\varphi'_{k-1}\alpha_k\varphi_k.$$

Cancelling $\varphi'_1\cdots\varphi'_{k-1}$ on the left, we have that $\varphi'_k\alpha_{k+1} = \alpha_k\varphi_k$. In the same way, it follows that $\alpha_d\varphi_d = \varphi'_d\alpha_1$ and therefore the d-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ forms a morphism $X \to X'$. Since $\Psi_d(\alpha) = \beta$ by construction, the functor Ψ_d is full.

Next, we show that Ψ_d is faithful. Suppose $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathrm{Hom}_{\mathrm{MF}^d_S(f)}(X, X')$ such that $\Psi_d(\alpha) = 0$. Then there exist homomorphisms $s_{j+1} : F_1 \to F'_{j+1}$ such that

$$\varphi'_1\varphi'_2 \cdots \varphi'_j s_{j+1} = \alpha_1 \quad \text{and} \quad s_{j+1}\varphi_1\varphi_2 \cdots \varphi_j = \alpha_{j+1}$$

for each $1 \leq j \leq d-1$. In particular, $\alpha_1 = \varphi'_1 s_2$ and, for $2 \leq j \leq d-2$, we have that

$$\varphi'_1\varphi'_2 \cdots \varphi'_{j-1} s_j = \alpha_1 = \varphi'_1\varphi'_2 \cdots \varphi'_j s_{j+1}.$$

We may cancel $\varphi'_1\varphi_2 \cdots \varphi'_{j-1}$ on the left to obtain the equation $s_j = \varphi'_j s_{j+1}$ which holds for each $2 \leq j \leq d-2$. Finally, we have that

$$\varphi'_1\varphi'_2 \cdots \varphi'_{d-1} s_d \cdot f = f \cdot \alpha_1 = \varphi'_1\varphi'_2 \cdots \varphi'_{d-1}\varphi'_d \alpha_1$$

which implies that $s_d \cdot f = \varphi'_d \alpha_1$. Hence, we have a commutative diagram

$$\begin{array}{ccccccccccccc}
F_1 & \xrightarrow{\varphi_d} & F_d & \xrightarrow{\varphi_{d-1}} & F_{d-1} & \xrightarrow{\varphi_{d-2}} & \cdots & \xrightarrow{\varphi_3} & F_3 & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\varphi_1} & F_1 \\
\| & & \downarrow{\scriptstyle \Phi_{d-1}} & & \downarrow{\scriptstyle \Phi_{d-2}} & & & & \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \varphi_1} & & \| \\
F_1 & \xrightarrow{f \cdot 1_{F_1}} & F_1 & \xrightarrow{1_{F_1}} & F_1 & \xrightarrow{1_{F_1}} & \cdots & \xrightarrow{1_{F_1}} & F_1 & \xrightarrow{1_{F_1}} & F_1 & \xrightarrow{1_{F_1}} & F_1 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle s_d} & & \downarrow{\scriptstyle s_{d-1}} & & & & \downarrow{\scriptstyle s_3} & & \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle \alpha_1} \\
F'_1 & \xrightarrow{\varphi'_d} & F'_d & \xrightarrow{\varphi'_{d-1}} & F'_{d-1} & \xrightarrow{\varphi'_{d-2}} & \cdots & \xrightarrow{\varphi'_3} & F'_3 & \xrightarrow{\varphi'_2} & F'_2 & \xrightarrow{\varphi'_1} & F'_1.
\end{array}$$

The middle matrix factorization is isomorphic to a direct sum of $\operatorname{rank}_S(F_1)$ copies of $\mathcal{P}_d$. Thus, $\alpha = 0$ in $\operatorname{MF}^d_S(f)/\{\mathcal{P}_d\}$ and Ψ_d is faithful. □

Corollary 5.2.8. *The equivalence* $\Psi : \operatorname{MF}^d_S(f)/\{\mathcal{P}_d\} \to \mathcal{F}_{d-1}(R)$ *induces an equivalence between the stable category* $\underline{\operatorname{MF}}^d_S(f) = \operatorname{MF}^d_S(f)/\{\mathcal{P}_1, \ldots, \mathcal{P}_d\}$*, defined in Chapter 2, and* $\mathcal{F}_{d-1}(R)$ *modulo sequences of surjections consisting of only free* R*-modules.* □

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and let $d, m \geq 2$ be integers. Theorem 5.2.2 combined with Corollary 4.4.2 imply that the category $\mathcal{F}_{d-1}(\mathbf{k}[\![y]\!]/(y^m))$ has finite representation type if and only if $(d,m) \in \{(2,m),(3,3),(3,4),(3,5),(4,3),(5,3)\}$. This is a known result of Simson [Sim07, Theorem 3.5]. In this sense, Corollary 4.4.2 is an extension of Simson's results to the case of a hypersurface ring of higher dimension. Furthermore, we can give a slightly more general result, analogous to Corollary 4.4.2 in the context of this chapter.

Proposition 5.2.9. *Let $(R,\mathfrak{m},k)$ be a complete Gorenstein local ring containing k, an algebraically closed field of characteristic zero. Assume there are, up to isomorphism, only finitely many indecomposable objects in $\mathcal{F}_n(R)$ for some $n \geq 2$. Then R is isomorphic to a complete A_k singularity for $k = 1,2,3,$ or 4. That is,*

$$R \cong k[\![y, x_2, \ldots, x_r]\!]/(y^{k+1} + x_2^2 + \cdots + x_r^2)$$

for $k = 1,2,3,$ or 4.

Proof. Let $M \in \mathrm{MCM}(R)$ and let $\xi_M = \big(M \to 0 \to \cdots \to 0 \to 0\big) \in \mathcal{F}_n(R)$. Then there is an isomorphism of rings:

$$\mathrm{End}_{\mathcal{F}_n(R)}(\xi_M) \cong \mathrm{End}_R(M).$$

Since R is complete, the module M is indecomposable if and only if $\mathrm{End}_R(M)$ is a local ring. Thus, if M is indecomposable, then $\mathrm{End}_{\mathcal{F}_n(R)}(\xi_M)$ is local which implies that ξ_M is indecomposable as well. In other words, there are at least as many non-isomorphic indecomposable objects in $\mathcal{F}_n(R)$ as there are in $\mathrm{MCM}(R)$.

Now, assume there are only finitely many indecomposable objects in $\mathcal{F}_n(R)$ up to isomorphism. Then R has finite CM type by the above observation. It follows from [LW12, Theorem 9.16] that R is isomorphic to a complete ADE hypersurface singularity. By Theorem 5.2.2 and Corollary 4.4.2 the only hypersurface rings with the property that $\mathcal{F}_n(R)$ has finite type for some $n \geq 2$ are the ones isomorphic to an A_1, A_2, A_3, or A_4 hypersurface singularity. ∎

To end this section, we use Theorem 5.2.2 to elaborate on Example 4.4.4.

Example 5.2.10. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let $S = \mathbf{k}[\![y]\!]$, $f = y^4 \in S$, and $R = S/(f)$. In Example 4.4.4, we computed the complete set of 21 isomorphism classes of indecomposable 3-fold matrix factorizations of $f = y^4$. Each of these factorizations corresponds to an indecomposable object in $\mathcal{F}_2(R)$ (except $\mathcal{P}_3$ which corresponds to zero). Furthermore, Theorem 5.2.2 implies that Table 5.1 below contains a complete set of isomorphism classes of indecomposable objects in $\mathcal{F}_2(R)$. We use the same notation as Example 4.4.4.

Table 5.1: Indecomposable objects in $\mathcal{F}_2(R)$

X	$\Psi_3(X)$	$\Psi_3(TX)$	$\Psi_3(T^2X)$
$\mathcal{P}_1$	$R \xrightarrow{1} R$	$R \to 0$	$0 \to 0$
X_{φ_1}	$R \to R/(y^3)$	$R/(y) \xrightarrow{1} R/(y)$	$R/(y^3) \to 0$
X_{ψ_1}	$R/(y^3) \xrightarrow{1} R/(y^3)$	$R/(y) \to 0$	$R \to R/(y)$
X_{φ_2}	$R \to R/(y^2)$	$R/(y^2) \xrightarrow{1} R/(y^2)$	$R/(y^2) \to 0$
X_β	$R/(y^3) \to R/(y^2)$	$R/(y^2) \to R/(y)$	$R/(y^3) \to R/(y)$
X_α	$R \oplus R/(y) \to R/(y^2)$	$R \oplus R/(y^2) \to R/(y^3)$	$R \oplus R/(y) \to R/(y^3)$
X_ξ	$R/(y^2) \oplus R \to R/(y) \oplus R/(y^3)$	$R/(y^3) \oplus R/(y) \to R/(y^2)$	$R \oplus R/(y^2) \to R/(y^2)$

It is not hard to compute the surjections depicted above; with the exception of the surjections induced by X_ξ, they are the obvious maps. The maps induced by $X_\xi = (\varphi_1, \varphi_2, \varphi_3) \in \mathrm{MF}^3_S(y^4)$ and its shifts are also easy to compute. For instance, $\Psi_3(X_\xi)$ can be computed by the diagram below with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S^2 & \xrightarrow{\begin{pmatrix} y^2 & 0 \\ 0 & y^4 \end{pmatrix}} & S^2 & \longrightarrow & R/(y^2) \oplus R & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}} & & \Big\downarrow & & \\
0 & \longrightarrow & S^2 & \xrightarrow{\begin{pmatrix} 0 & y^2 \\ y^4 & y^3 \end{pmatrix}} & S^2 & \longrightarrow & \operatorname{cok}(\varphi_1\varphi_2) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix}} & & \Big\| & & \Big\downarrow & & \\
0 & \longrightarrow & S^2 & \xrightarrow[\begin{pmatrix} y & 0 \\ 0 & y^3 \end{pmatrix}]{} & S^2 & \longrightarrow & R/(y) \oplus R/(y^3) & \longrightarrow & 0.
\end{array}
$$

Thus, the map $R/(y^2) \oplus R \to R/(y) \oplus R/(y^3)$ is given by $(\overline{a}, b) \mapsto (\overline{a}, \overline{ya+b})$ where $a, b \in R$ and $\overline{(-)}$ indicates the image of a or b in the appropriate quotient.

6 | Tensor Products of Matrix Factorizations

Let $\mathbf{k}$ be a field. In [Yos98], Yoshino introduces a construction which he refers to as the tensor product of matrix factorizations. Namely, given a (2-fold) matrix factorization $X = (\varphi : G \to F, \psi : F \to G)$ of an element $f \in S_1 = \mathbf{k}[\![x_1, \dots, x_r]\!]$ and another (2-fold) factorization $Y = (\varphi' : G' \to F', \psi' : F' \to G')$ of $g \in S_2 = \mathbf{k}[\![y_1, \dots, y_s]\!]$, the tensor product $X \widehat{\otimes} Y$ is a matrix factorization of $f + g \in S = \mathbf{k}[\![x_1, \dots, x_r, y_1, \dots, y_s]\!]$ given by the formula

$$X \widehat{\otimes} Y := \left(\begin{pmatrix} \varphi \otimes 1_{G'} & 1_G \otimes \varphi' \\ -1_F \otimes \psi' & \psi \otimes 1_{F'} \end{pmatrix}, \begin{pmatrix} \psi \otimes 1_{G'} & -1_F \otimes \varphi' \\ 1_G \otimes \psi' & \varphi \otimes 1_{F'} \end{pmatrix} \right) \in \mathrm{MF}^2_S(f+g).$$

This construction is a generalization of the functors introduced by Knörrer to study the relationship between a hypersurface and its double branched cover (see [Knö87, Section 2]).

In this chapter we build upon the work of Knörrer [Knö87] and Yoshino [Yos98] by defining a tensor product of matrix factorizations with d factors. To do so, we will use a construction given by Bläser-Eisenbud-Schreyer [BES17, Proposition 2.1]. We also study basic properties of the construction and provide some criteria for when it preserves indecomposability.

6.1 Definition

Throughout this chapter we will use the following notation.

Notation 6.1.1. Let $\mathbf{k}$ be a field and fix an integer $d \geq 2$. Let $S_1 = \mathbf{k}[\![x]\!] = \mathbf{k}[\![x_1, x_2, \dots, x_r]\!]$,

$S_2 = \mathbf{k}[\![y]\!] = \mathbf{k}[\![y_1, y_2, \ldots, y_s]\!]$, and $S = \mathbf{k}[\![x, y]\!] = \mathbf{k}[\![x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s]\!]$. Fix non-zero non-invertible elements $f \in S_1$ and $g \in S_2$. Let $X = (\varphi_1 : F_2 \to F_1, \varphi_2 : F_3 \to F_2, \ldots, \varphi_d : F_1 \to F_d) \in \mathrm{MF}^d_{S_1}(f)$ be of size n and $Y = (\psi_1 : G_2 \to G_1, \psi_2 : G_3 \to G_2, \ldots, \psi_d : G_1 \to G_d) \in \mathrm{MF}^d_{S_2}(g)$ be of size m. Assume $z_1, z_2, \ldots, z_d$ are elements of $\mathbf{k}$ such that $\prod_{j=1}^{d}(z - z_j) = z^d - a$ where $a = 1$ if d is odd and $a = -1$ if d is even.

For a homomorphism of S-modules $g : M \to N$ and finite direct sum decompositions, $M = \bigoplus_j M_j$ and $N = \bigoplus_i N_i$, we let $g(i, j) : M_j \to N_i$ denote the ij component of g with respect to the given direct sum decompositions.

Definition 6.1.2. For each $k \in \mathbb{Z}_d$, use the same symbols $\varphi_k : F_{k+1} \to F_k$, respectively $\psi_k : G_{k+1} \to G_k$, to denote the induced homomorphisms $\varphi_k \otimes 1_S : F_{k+1} \otimes_{S_1} S \to F_k \otimes_{S_1} S$, respectively $\psi_k \otimes 1_S : G_{k+1} \otimes_{S_2} S \to G_k \otimes_{S_2} S$. The amounts to considering φ_i and ψ_j as matrices over S after picking bases. For each $k \in \mathbb{Z}_d$, let $\mathcal{F}_k = \bigoplus_{j=1}^{d}(F_{k+j-1} \otimes_S G_{2-j})$ which is a free S-module of rank dnm. Then, for $k \in \mathbb{Z}_d$, we define a homomorphism $\Phi_k : \mathcal{F}_{k+1} \to \mathcal{F}_k$

$$\Phi_k := \begin{pmatrix} \varphi_k \otimes 1_{G_1} & 0 & 0 & 0 & \ldots & z_k 1_{F_k} \otimes \psi_1 \\ z_k 1_{F_{k+1}} \otimes \psi_d & \varphi_{k+1} \otimes 1_{G_d} & 0 & 0 & \ldots & 0 \\ 0 & z_k 1_{F_{k+2}} \otimes \psi_{d-1} & \varphi_{k+2} \otimes 1_{G_{d-1}} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & z_k 1_{F_{k-1}} \otimes \psi_2 & \varphi_{k-1} \otimes 1_{G_2} \end{pmatrix}.$$

Let $X \widehat{\otimes} Y := (\Phi_1, \Phi_2, \ldots, \Phi_d)$ be the *tensor product* of X and Y with respect to $z_1, z_2, \ldots, z_d$.

Proposition 6.1.3. *For any $X \in \mathrm{MF}^d_{S_1}(f)$ of size n and $Y \in \mathrm{MF}^d_{S_2}(g)$ of size m, the tensor product $X \widehat{\otimes} Y$ is a matrix factorization of $f + g$ of size dnm.*

Proof. Since X is of size n we may assume $\varphi_i : F \to F$, $i \in \mathbb{Z}_d$, where $F = S_1^n$. Similarly, since Y is of size m, we assume $\psi_j : G \to G$, $j \in \mathbb{Z}_d$, where $G = S_2^m$. Let $X \widehat{\otimes} Y = (\Phi_1 : (F \otimes_S G)^d \to (F \otimes_S G)^d, \ldots, \Phi_d : (F \otimes_S G)^d \to (F \otimes_S G)^d)$ as in 6.1.2. We set $A_i = \varphi_i \otimes_S 1_G$

and $B_j = 1_F \otimes \psi_{1-j}$ for $i, j \in \mathbb{Z}_d$ and apply [BES17, Proposition 2.2] to see that

$$\begin{aligned}\Phi_1\Phi_2\cdots\Phi_d &= \big(A_1\cdots A_d + (-1)^{d+1}aB_d\cdots B_1\big)\cdot 1_{(F\otimes_S G)^d}\\ &= \big(f(1_F\otimes 1_G) + (-1)^{d+1}ag(1_F\otimes 1_G)\big)\cdot 1_{(F\otimes_S G)^d}.\end{aligned}$$

The statement and proof of [BES17, Proposition 2.2] is given in the case $a = 1$ but the proof works equally well for $a = -1$. By assumption, if d is odd, then $a = 1$ and $(-1)^{d+1}a = 1$. If d is even, then $a = -1$ and $(-1)^{d+1}a = 1$. In either case we get that $\Phi_1\cdots\Phi_d = (f+g)\cdot 1_{(F\otimes_S G)^d}$, that is, $(\Phi_1,\ldots,\Phi_d) \in \mathrm{MF}^d_S(f+g)$ as desired. Since $F\otimes_S G$ is of rank nm, the matrix factorization $X\widehat{\otimes}Y$ is of size dnm. □

Lemma 6.1.4. *For $Y \in \mathrm{MF}^d_{S_2}(g)$, the tensor product $(-)\widehat{\otimes}Y$ defines a functor $\mathrm{MF}^d_{S_1}(f) \to \mathrm{MF}^d_S(f+g)$. Similarly, for $X \in \mathrm{MF}^d_{S_1}(f)$, $X\widehat{\otimes}(-)$ defines a functor $\mathrm{MF}^d_{S_2}(g) \to \mathrm{MF}^d_S(f+g)$.*

Proof. If $\alpha : X \to X'$ is a morphism in $\mathrm{MF}^d_{S_1}(f)$, where $X' = (\varphi'_2 : F'_2 \to F'_1, \ldots, \varphi'_d : F'_1 \to F'_d)$, then for $Y \in \mathrm{MF}^d_{S_2}(g)$ we define $\alpha\widehat{\otimes}1_Y$ in the following way: For each $k \in \mathbb{Z}_d$, let

$$(\alpha\widehat{\otimes}1_Y)_k = \begin{pmatrix} \alpha_k\otimes 1_{G_1} & 0 & \cdots & 0\\ 0 & \alpha_{k+1}\otimes 1_{G_d} & & 0\\ \vdots & & \ddots & \vdots\\ 0 & \cdots & & \alpha_{k-1}\otimes 1_{G_2}\end{pmatrix} : \mathcal{F}_k \to \mathcal{F}'_k,$$

where $\mathcal{F}_k$ is as in (6.1.2) and $\mathcal{F}'_k = \bigoplus_{j=1}^d (F'_{k+j-1}\otimes_S G_{2-j})$. Then

$$\alpha\widehat{\otimes}1_Y = ((\alpha\widehat{\otimes}1_Y)_1, \ldots, (\alpha\widehat{\otimes}1_Y)_d) : X\widehat{\otimes}Y \to X'\widehat{\otimes}Y$$

forms a morphism of matrix factorizations in $\mathrm{MF}^d_S(f+g)$.

Similarly, if $\beta : Y \to Y'$ is a morphism in $\mathrm{MF}^d_{S_2}(g)$ and $X \in \mathrm{MF}^d_{S_1}(f)$, then we define

$1_X \widehat{\otimes} \beta$ by setting

$$(1_X \widehat{\otimes} \beta)_k = \begin{pmatrix} 1_{F_{k+1}} \otimes \beta_1 & 0 & \cdots & 0 \\ 0 & 1_{F_{k+2}} \otimes \beta_d & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1_{F_k} \otimes \beta_2 \end{pmatrix} : \mathcal{F}_k \to \mathcal{F}''_k,$$

for each $k \in \mathbb{Z}_d$. Here $\mathcal{F}''_k = \bigoplus_{j=1}^{d} (F_{k+j-1} \otimes_S G'_{2-j})$. Then

$$1_X \widehat{\otimes} \beta = ((1_X \widehat{\otimes} \beta)_1, \ldots, (1_X \widehat{\otimes} \beta)_d) : X \widehat{\otimes} Y \to X \widehat{\otimes} Y'$$

forms a morphism of matrix factorizations in $\mathrm{MF}_S^d(f+g)$. □

Remark 6.1.5. If $d = 2$, then the functors $(-)\widehat{\otimes} Y$ and $X \widehat{\otimes}(-)$ are naturally isomorphic to the functors defined by Yoshino. In particular, these functors are a further generalization of the original functors defined by Knörrer (see [Yos98, Remark 1.3]).

Example 6.1.6. Consider the polynomial $f = xyz + uvw \in S = \mathbf{k}[\![x, y, z, u, v, w]\!]$. Assume there exists a primitive 3rd root of unity $\xi \in \mathbf{k}$. Then we have a matrix factorization

$$(x, y, z) \widehat{\otimes} (u, v, w) = \left(\begin{pmatrix} x & 0 & u \\ w & y & 0 \\ 0 & v & z \end{pmatrix}, \begin{pmatrix} y & 0 & \xi u \\ \xi w & z & 0 \\ 0 & \xi v & x \end{pmatrix}, \begin{pmatrix} z & 0 & \xi^2 u \\ \xi^2 w & x & 0 \\ 0 & \xi^2 v & y \end{pmatrix} \right) \in \mathrm{MF}_S^3(f).$$

Here, the tensor product is with respect to the ordering $\{1, \xi, \xi^2\}$. It will follow from Theorem 6.3.6 that this matrix factorization is indecomposable.

In the remaining sections, we will utilize the permutation matrix

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

to cyclically permute the columns and rows of matrices. It will be convenient to abuse notation and multiply by C on both the left and right of $d \times d$ block matrices as well as use C to indicate the permutation of a given direct sum decomposition. More precisely, given a direct sum of modules $H = H_1 \oplus H_2 \oplus \cdots \oplus H_d$ we let $C : H \to CH$ be the homomorphism $(h_1, h_2, \dots, h_d) \mapsto (h_d, h_1, \dots, h_{d-1})$, for $h_i \in H_i$, where $CH = H_d \oplus H_1 \oplus \cdots \oplus H_{d-1}$. Furthermore, for any $k \in \mathbb{Z}_d$, we have $C^k : H \to C^k H$ where $C^k H = \bigoplus_{j=1}^d H_{j-k}$.

Let $h : H \to H'$ be a homomorphism where $H = \bigoplus_{j=1}^d H_j$ and $H' = \bigoplus_{i=1}^d H'_i$. We conjugate h by C in the following sense: For $k \in \mathbb{Z}_d$, we have the composition

$$C^{-k} H \xrightarrow{C^k} H \xrightarrow{h} H' \xrightarrow{C^{-k}} C^{-k} H'.$$

The ij component of this composition, with respect to the specified direct sum decompositions, is the $(i+k)(j+k)$ component of the homomorphism h. That is,

$$C^{-k} h C^k(i,j) = h(i+k, j+k) : H_{j+k} \to H'_{i+k}.$$

The main convenience of this notation comes in describing $X \widehat{\otimes} Y$: Let X and Y be as above. For $k \in \mathbb{Z}_d$, let

$$A_k = \bigoplus_{j=1}^{d} (\varphi_{k+j-1} \otimes 1_{G_{2-j}}) : \mathcal{F}_{k+1} \to \mathcal{F}_k$$

and

$$B_k = \bigoplus_{j=1}^{d} (1_{F_{k+j}} \otimes \psi_{1-j}) : \mathcal{F}_{k+1} \to C^{-1}\mathcal{F}_k.$$

Then $\Phi_k = A_k + z_k C B_k$ so that

$$X \widehat{\otimes} Y = (A_1 + z_1 C B_1, A_2 + z_2 C B_2, \ldots, A_d + z_d C B_d)\,. \tag{6.1.1}$$

6.2 Basic properties

Before proceeding we record some functorial properties of $(-)\widehat{\otimes}(-)$ which we will use below. The first lemma, which follows by performing straightforward row and column operations, shows that $(-)\widehat{\otimes}(-)$ is additive in both components.

Lemma 6.2.1. *Let $X, X' \in \mathrm{MF}^d_{S_1}(f)$ and $Y \in \mathrm{MF}^d_{S_2}(g)$. There is an isomorphism $(X \oplus X')\widehat{\otimes} Y \cong (X \widehat{\otimes} Y) \oplus (X' \widehat{\otimes} Y)$. Similarly, for $Y, Y' \in \mathrm{MF}^d_{S_2}(f)$ and $X \in \mathrm{MF}^d_{S_1}(f)$, there is an isomorphism $X \widehat{\otimes}(Y \oplus Y') \cong (X \widehat{\otimes} Y) \oplus (X \widehat{\oplus} Y')$.* □

Lemma 6.2.2. *For any $i \in \mathbb{Z}_d$, there is an isomorphism $T^i X \widehat{\otimes} T^{-i} Y \cong X \widehat{\otimes} Y$*

Proof. Let $X = (\varphi_1 : F \to F, \ldots, \varphi_d : F \to F) \in \mathrm{MF}^d_{S_1}(f)$, where $F = S_1^n$, and $Y = (\psi_1 : G \to G, \ldots, \psi_d : G \to G) \in \mathrm{MF}^d_{S_2}(g)$, where $G = S_2^m$. Let A be the block diagonal matrix with $(\varphi_1 \otimes 1_G, \varphi_2 \otimes 1_G, \ldots, \varphi_d \otimes 1_G)$ down the diagonal. Similarly let B be the block diagonal matrix with $(1_F \otimes \psi_d, 1_F \otimes \psi_{d-1}, \ldots, 1_F \otimes \psi_1)$ down the diagonal. Since $C^{-k} A C^k (i,i) = A(i+k, i+k)$ for each $i, k \in \mathbb{Z}_d$, we have that $A_k = C^{-k+1} A C^{k-1}$. In other words, conjugation by C cyclically permutes the diagonal blocks of A. Using (6.1.1), we have

$$X \widehat{\otimes} Y = (A + z_1 C B, C^{-1} A C + z_2 C B, C^{-2} A C^2 + z_3 C B, \ldots, C^{1-d} A C^{d-1} + z_d C B).$$

Let $i, k \in \mathbb{Z}_d$. Then $T^i X = (\varphi_{i+1}, \varphi_{i+2}, \ldots, \varphi_{i-1}, \varphi_i)$ and $T^{-i} Y = (\psi_{1-i}, \psi_{2-i}, \ldots, \psi_{-1-i}, \psi_{-i})$.

In particular, $T^iX\widehat{\otimes}T^{-i}Y = (\Phi'_1, \Phi'_2, \ldots, \Phi'_d)$, where

$$\Phi'_k = \begin{pmatrix} \varphi_{i+k}\otimes 1_G & 0 & 0 & 0 & \cdots & z_k 1_F\otimes\psi_{1-i} \\ z_k 1_F\otimes\psi_{-i} & \varphi_{i+k+1}\otimes 1_G & 0 & 0 & \cdots & 0 \\ 0 & z_k 1_F\otimes\psi_{-1-i} & \varphi_{i+k+2}\otimes 1_G & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & z_k 1_F\otimes\psi_{2-i} & \varphi_{i+k-1}\otimes 1_{G_{2-i}} \end{pmatrix}.$$

Again, since conjugation by C cyclically permutes the diagonal blocks, we have that

$$\begin{aligned} \Phi'_k &= C^{-i}(C^{-k+1}AC^{k-1})C^i + z_k C(C^{-i}BC^i) \\ &= C^{-i-k+1}AC^{i+k-1} + z_k C^{1-i}BC^i. \end{aligned}$$

Furthermore, notice that

$$\begin{aligned} C^{-i}(C^{-k+1}AC^{k-1} + z_k CB) &= C^{-i-k+1}AC^{k-1} + z_k C^{1-i}B \\ &= (C^{-i-k+1}AC^{i+k-1} + z_k C^{1-i}BC^i)C^{-i}. \end{aligned}$$

This implies that we have a commutative diagram with vertical isomorphisms:

$$\begin{array}{ccccccccc} (F\otimes G)^d & \xrightarrow{\Phi_d} & (F\otimes G)^d & \xrightarrow{\Phi_{d-1}} & \cdots & \xrightarrow{\Phi_2} & (F\otimes G)^d & \xrightarrow{\Phi_1} & (F\otimes G)^d \\ \downarrow{\scriptstyle C^{-i}} & & \downarrow{\scriptstyle C^{-i}} & & & & \downarrow{\scriptstyle C^{-i}} & & \downarrow{\scriptstyle C^{-i}} \\ (F\otimes G)^d & \xrightarrow{\Phi'_d} & (F\otimes G)^d & \xrightarrow{\Phi'_{d-1}} & \cdots & \xrightarrow{\Phi'_2} & (F\otimes G)^d & \xrightarrow{\Phi'_1} & (F\otimes G)^d. \end{array}$$

Hence, $X\widehat{\otimes}Y \cong T^iX\widehat{\otimes}T^{-i}Y$ as desired. □

Definition 6.2.3. If $(\Psi_1 : H_2 \to H_1, \ldots, \Psi_d : H_1 \to H_d) \in \mathrm{MF}^d_S(f+g)$, then the d-tuple of homomorphisms $\big(\Psi_1 \otimes_S 1_{S/(y)}, \ldots, \Psi_d \otimes_S 1_{S/(y)}\big)$ forms a matrix factorization of f, where now the k-th map is a $S/(y) \cong S_1$-homomorphism $H_{k+1} \otimes_S S/(y) \to H_k \otimes S/(y)$. Similarly, for a morphism $(\alpha_1, \ldots, \alpha_d) \in \mathrm{MF}^d_S(f+g)$, the tuple $(\alpha_1 \otimes_S 1_{S/(y)}, \ldots, \alpha_d \otimes_S 1_{S/(y)})$ forms a morphism in $\mathrm{MF}^d_{S_1}(f)$. Thus, we have defined a functor $\mathrm{MF}^d_S(f+g) \to \mathrm{MF}^d_{S_1}(f)$. Following

[Yos98], we call this functor reduction to $S_1 \cong S/(y)$ and denote it $(-)_y$. In the same way, we define the functor reduction to $S_2 \cong S/(x)$: $(-)_x : \mathrm{MF}^d_S(f+g) \to \mathrm{MF}^d_{S_2}(g)$.

Lemma 6.2.4.

(i) Suppose $Y \in \mathrm{MF}^d_{S_2}(g)$ is reduced and of size m. Then there is an isomorphism

$$(X \widehat{\otimes} Y)_y \cong X^m \oplus (TX)^m \oplus (T^2X)^m \oplus \cdots \oplus (T^{d-1}X)^m.$$

(ii) Suppose $X \in \mathrm{MF}^d_{S_1}(f)$ is reduced and of size n. Then there is an isomorphism

$$(X \widehat{\otimes} Y)_x \cong (Y)^n \oplus (T^{d-1}Y)^n \oplus (T^{d-2}Y)^n \oplus \cdots \oplus (TY)^n.$$

Proof. We prove only (ii) since (i) follows directly from 6.1.2. Using the notation of (6.1.1), we have that $X \widehat{\otimes} Y = (\Phi_1, \Phi_2, \ldots, \Phi_d)$ where $\Phi_k = A_k + z_k C B_k$ for each $k \in \mathbb{Z}_d$. We have a commutative diagram of S-modules where each vertical map is an isomorphism:

$$\begin{array}{ccccccccccc}
\mathcal{F}_1 & \xrightarrow{z_d C B_d} & \mathcal{F}_d & \xrightarrow{z_{d-1} C B_{d-1}} & \cdots & \xrightarrow{z_3 C B_3} & \mathcal{F}_3 & \xrightarrow{z_2 C B_2} & \mathcal{F}_2 & \xrightarrow{z_1 C B_1} & \mathcal{F}_1 \\
\downarrow{\scriptstyle 1_{\mathcal{F}_1}} & & \downarrow{\scriptstyle z_1 \cdots z_{d-1} C^{d-1}} & & & & \downarrow{\scriptstyle z_1 z_2 C^2} & & \downarrow{\scriptstyle z_1 C} & & \downarrow{\scriptstyle 1_{\mathcal{F}_1}} \\
\mathcal{F}_1 & \xrightarrow[B_d]{} & C^{d-1}\mathcal{F}_d & \xrightarrow[C^{-1}B_{d-1}C]{} & \cdots & \xrightarrow[C^3 B_3 C^{-3}]{} & C^2\mathcal{F}_3 & \xrightarrow[C^2 B_2 C^{-2}]{} & C\mathcal{F}_2 & \xrightarrow[C B_1 C^{-1}]{} & \mathcal{F}_1.
\end{array}$$

We claim that, after reduction to $S/(x)$, this diagram will give us the desired isomorphism of matrix factorizations.

Since X is reduced, $\varphi_k \otimes 1_{S/(x)} = 0$ for all $k \in \mathbb{Z}_d$ and therefore $A_k \otimes 1_{S/(x)} = 0$, also for each $k \in \mathbb{Z}_d$. Therefore,

$$(X \widehat{\otimes} Y)_x = (z_1 C B_1 \otimes 1_{S/(x)}, z_2 C B_2 \otimes 1_{S/(x)}, \ldots, z_d C B_d \otimes 1_{S/(x)}).$$

Since the free S-modules $F_1, \ldots, F_d$ are of rank n, we identify $(1_{F_i} \otimes \psi_j) \otimes 1_{S/(x)}$ with ψ_j^n, the direct sum of n copies of ψ_j, for all $i, j \in \mathbb{Z}_d$. For each $k \in \mathbb{Z}_d$, consider the conjugate

$C^k B_k C^{-k}$ of B_k. Since $B_k = \bigoplus_{j=1}^d (1_{F_{j+k}} \otimes \psi_{1-j})$, we have that

$$(C^k B_k C^{-k})_x := C^k B_k C^{-k} \otimes 1_{S/(x)} = \bigoplus_{j=1}^{d} \psi^n_{1-j+k}.$$

Therefore,

$$\begin{aligned} ((CB_1C^{-1})_x, (C^2B_2C^{-2})_x, \ldots, (B_d)_x) &= \left(\bigoplus_{j=1}^{d} \psi^n_{2-j}, \bigoplus_{j=1}^{d} \psi^n_{3-j}, \ldots, \bigoplus_{j=1}^{d} \psi^n_{1-j} \right) \\ &= \bigoplus_{j=1}^{d} (\psi^n_{2-j}, \psi^n_{3-j}, \ldots, \psi^n_{1-j}) \\ &= \bigoplus_{j=1}^{d} (T^{1-j}Y)^n. \end{aligned}$$

Thus, tensoring the original diagram with $S/(x)$ induces an isomorphism of matrix factorizations $(X \widehat{\otimes} Y)_x \cong \bigoplus_{j=1}^d (T^{1-j}Y)^n$ as desired. □

Recall that f has finite d-MF type if the category $\mathrm{MF}^d_S(f)$ has only finitely many non-isomorphic indecomposable objects. From Lemma 6.2.4, we have a generalization of one direction of Theorem 4.3.7.

Proposition 6.2.5. *Suppose $g \in S_2$ is a monomial of degree at least d. Then, for any $X \in \mathrm{MF}^d_{S_1}(f)$, there exists $Z \in \mathrm{MF}^d_S(f+g)$ such that X is isomorphic to a direct summand of Z_y. In particular, if $f+g$ has finite d-MF type, then so does f.*

Proof. Since g is a monomial in $y_1, \ldots, y_s$ of degree at least d, there exists a reduced matrix factorization $Y \in \mathrm{MF}^d_{S_2}(g)$ of size 1. For any $X \in \mathrm{MF}^d_{S_1}(f)$, Lemma 6.2.4(i) implies that X is isomorphic to a direct summand of $(X \widehat{\otimes} Y)_y$. Proceding as in the proof of Theorem 4.3.7, the result follows. □

6.2.1 Morphisms between tensor products

In the next section we will investigate the number of indecomposable summands of $X\widehat{\otimes}Y$. To do so, we will work directly with idempotents in the endomorphism ring of $X\widehat{\otimes}Y$. Here, we introduce the notation needed to keep track of morphisms between tensor products of matrix factorizations and we show how the functors $(-)_x$ and $(-)_y$ interact with morphisms of this form.

Let X and Y be as in (6.1.1), $X' = (\varphi'_1 : F'_2 \to F'_1, \ldots, \varphi'_d : F'_1 \to F'_d) \in \mathrm{MF}^d_{S_1}(f)$, and $Y' = (\psi'_1 : G'_2 \to G'_1, \ldots, \psi'_d : G'_1 \to G'_d) \in \mathrm{MF}^d_{S_2}(g)$.

Lemma 6.2.6. *Assume Y is reduced of size m and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) : X\widehat{\otimes}Y \to X'\widehat{\otimes}Y$ be a morphism in $\mathrm{MF}^d_S(f+g)$. Set $X'\widehat{\otimes}Y = (\Phi'_1 : \mathcal{F}'_2 \to \mathcal{F}'_1, \ldots, \Phi'_d : \mathcal{F}'_1 \to \mathcal{F}'_d)$, where $\mathcal{F}'_k = \bigoplus_{j=1}^d (F'_{k+j-1} \otimes G_{2-j})$, $k \in \mathbb{Z}_d$. For each $k \in \mathbb{Z}_d$, $\alpha_k = (\alpha_k(i,j))_{i,j\in\mathbb{Z}_d} : \mathcal{F}_k \to \mathcal{F}'_k$ where $\alpha_k(i,j) : F_{k+j-1} \otimes_S G_{2-j} \to F'_{k+i-1} \otimes_S G_{2-i}$ is a homomorphism of free S-modules. Then, after reduction to $S/(y)$, we have a morphism*

$$\alpha_y : \bigoplus_{q=1}^{d} \left(T^{q-1}X\right)^m \to \bigoplus_{p=1}^{d} \left(T^{p-1}X'\right)^m,$$

and the pq component of α_y with respect to this direct sum decomposition is the morphism

$$(\alpha_1(p,q), \alpha_2(p,q), \ldots, \alpha_d(p,q))_y : (T^{q-1}X)^m \to (T^{p-1}X')^m.$$

Proof. By Lemma 6.2.4(i), we have that $(X\widehat{\otimes}Y)_y = \left(\bigoplus_{j=1}^d \varphi_j^m, \bigoplus_{j=1}^d \varphi_{j+1}^m, \ldots, \bigoplus_{j=1}^d \varphi_{j-1}^m\right)$ and $(X'\widehat{\otimes}Y)_y = \left(\bigoplus_{j=1}^d (\varphi'_j)^m, \bigoplus_{j=1}^d (\varphi'_{j+1})^m, \ldots, \bigoplus_{j=1}^d (\varphi'_{j-1})^m\right)$. Since $(\alpha_k)_y(i,j) = (\alpha_k(i,j))_y : (F_{k+j-1} \otimes_S G_{2-j})_y \to (F'_{k+i-1} \otimes_S G_{2-j})_y$ for each $i,j,k \in \mathbb{Z}_d$, the morphism $\alpha_y : (X\widehat{\otimes}Y)_y \to (X'\widehat{\otimes}Y)_y$ forces that $\alpha_k(i,j)_y \varphi^m_{j+k-1} = (\varphi'_{i+k-1})^m \alpha_{k+1}(i,j)_y$. Hence, we have a morphism $(\alpha_1(i,j)_y, \alpha_2(i,j)_y, \ldots, \alpha_d(i,j)_y)$ for all $i,j,k \in \mathbb{Z}_d$.

The pq component of α_y is given by the composition

$$(T^{q-1}X)^m \hookrightarrow \bigoplus_{q=1}^{d} \left(T^{q-1}X\right)^m \xrightarrow{\alpha_y} \bigoplus_{p=1}^{d} \left(T^{p-1}X'\right)^m \twoheadrightarrow (T^{p-1}X')^m$$

where the left most map is the natural inclusion and the right most map is the natural projection. It follows that $\alpha_y(p,q) = (\alpha_1(p,q)_y, \alpha_2(p,q)_y, \dots, \alpha_d(p,q)_y)$. □

The tensor product $X\widehat{\otimes}Y$ puts the matrix factorization Y on the sub-diagonal (mod d) of the block matrices Φ_k. For this reason, reduction mod x of a morphism $\beta : X\widehat{\otimes}Y \to X\widehat{\otimes}Y'$ causes a "mixing" of the components of β. This is in contrast with Lemma 6.2.4(i) where we saw that reduction mod y is "diagonal". We make this observation precise in the next lemma.

Lemma 6.2.7. *Assume X is reduced of size n and let $\beta = (\beta_1, \beta_2, \dots, \beta_d) : X\widehat{\otimes}Y \to X\widehat{\otimes}Y'$ be a morphism in $\mathrm{MF}_S^d(f+g)$. Set $X\widehat{\otimes}Y' : (\Phi_1' : \mathcal{F}_2' \to \mathcal{F}_1', \dots, \Phi_d' : \mathcal{F}_1' \to \mathcal{F}_d')$, where $\mathcal{F}_k' = \bigoplus_{j=1}^{d}(F_{k+j-1} \otimes G_{2-j}')$, $k \in \mathbb{Z}_d$. For each $k \in \mathbb{Z}_d$, $\beta_k = (\beta_k(i,j))_{i,j\in\mathbb{Z}_d} : \mathcal{F}_k \to \mathcal{F}_k'$, where $\beta_k(i,j) : F_{k+j-1} \otimes_S G_{2-j} \to F_{k+i-1} \otimes_S G_{2-i}'$ is a homomorphism of free S-modules. Then, after reduction to $S/(x)$, we have a morphism*

$$\tilde{\beta}_x : \bigoplus_{q=1}^{d} \left(T^{1-q}Y\right)^n \to \bigoplus_{p=1}^{d} \left(T^{1-p}Y'\right)^n,$$

induced by β_x, and the pq component of $\tilde{\beta}_x$ with respect to this direct sum decomposition is the morphism

$$(\beta_1(p,q), \beta_2(p-1,q-1), \dots, \beta_d(p+1,q+1))_x : (T^{1-q}Y)^n \to (T^{1-p}Y')^n.$$

Proof. By Lemma 6.2.4(ii), there are isomorphisms $\xi_1 : (X\widehat{\otimes}Y)_x \to \bigoplus_{p=1}^{d}(T^{1-p}Y)^n$ and $\xi_2 : (X\widehat{\otimes}Y')_x \to \bigoplus_{q=1}^{d}(T^{1-q}Y')^n$. In particular, $\xi_1 = (1_{\mathcal{F}_1}, z_1C, z_1z_2C^2, \dots, z_1\cdots z_{d-1}C^{d-1})$

and $\xi_2 = (1_{\mathcal{F}_1'}, z_1C, z_1z_2C^2, \dots, z_1 \cdots z_{d-1}C^{d-1})$. Set $\tilde{\beta}_x = \xi_2\beta_x\xi_1^{-1}$. Then we have that

$$\tilde{\beta}_x = \big((\beta_1)_x, C(\beta_2)_xC^{-1}, C^2(\beta_3)_xC^{-2}, \dots, C^{d-1}(\beta_d)_xC^{-d+1}\big)$$

since $\xi_1^{-1} = (1_{\mathcal{F}_1}, z_1^{-1}C^{-1}, \dots, (z_1 \cdots z_d)^{-1}C^{-d+1})$.

Finally, the pq component of $\tilde{\beta}_x : \bigoplus_{q=1}^{d} (T^{1-q}Y)^n \to \bigoplus_{p=1}^{d} (T^{1-p}Y')^n$ with respect to the given direct sum decomposition is

$$\begin{aligned}\tilde{\beta}_x(p,q) &= \big((\beta_1)_x(p,q), C(\beta_2)_xC^{-1}(p,q), \dots, C^{d-1}(\beta_d)_xC^{-d+1}(p,q)\big)\\ &= ((\beta_1)_x(p,q), (\beta_2)_x(p-1,q-1), \dots, (\beta_d)_x(p+1,q+1))\\ &= (\beta(p,q)_x, \beta_2(p-1,q-1)_x, \dots, \beta_d(p+1,q+1)_x)\,.\end{aligned}$$

□

6.3 Decomposability of tensor products

In this section, we investigate the number of indecomposable summands of $X\widehat{\otimes}Y$. Many of the results we present are extensions of ones found in [Yos98, §3]. In particular, we will consider the tensor product $X\widehat{\otimes}Y$ in the extremal cases with respect to the order of X and Y. In other words, we will consider the case when $X \cong TX$ and $Y \cong TY$ (Proposition 4.5.4) and the case when both X and Y have order d (Theorem 6.3.6).

We continue using the notations 6.1.1. Our first result utilizes the following polynomial identity:

Lemma 6.3.1. *Let A and B be commuting variables and assume $\omega \in \mathbf{k}$ is a primitive d-th root of* 1. *Then*

$$(A+B)(A+\omega B)(A+\omega^2 B)\cdots(A+\omega^{d-1}B) = A^d + (-1)^{d+1}B^d.$$

□

6.3.1 The case of order one

Proposition 6.3.2. *Assume* $\mathbf{k}$ *is an algebraically closed field with* $\operatorname{char}\mathbf{k}$ *not dividing* d*, let* $\omega \in \mathbf{k}$ *be a primitive* d*-th root of* 1*, and let* $\mu \in \mathbf{k}$ *be a* d*-th root of* -1*. Suppose* $X \cong TX$ *and* $Y \cong TY$*. Then there exists* $Z \in \mathrm{MF}_S^d(f+g)$ *such that*

$$X \widehat{\otimes} Y \cong \bigoplus_{j \in \mathbb{Z}_d} T^j(Z)$$

where the tensor product is taken with respect to $z_1, z_2, \ldots, z_d$ *where*

$$z_k = \begin{cases} \omega^{k-1} & \text{if } d \text{ is odd} \\ \mu\omega^{k-1} & \text{if } d \text{ is even} \end{cases}.$$

Proof. By Proposition 4.5.2, we may assume $X = (\varphi, \varphi, \ldots, \varphi) \in \mathrm{MF}_{S_1}^d(f)$ for some $\varphi : F \to F$ such that $\varphi^d = f \cdot 1_F$ and $Y = (\psi, \psi, \ldots, \psi) \in \mathrm{MF}_{S_2}^d(g)$ for some $\psi : G \to G$ such that $\psi^d = g \cdot 1_G$. If d is odd, set $B = 1_F \otimes_S \psi$ while if d is even, set $B = \mu(1_F \otimes_S \psi)$. In either case, take $A = \varphi \otimes_S 1_G$. Since $AB = BA$, $A^d = f \cdot 1_{F \otimes G}$, and $(-1)^{d+1} B^d = g \cdot 1_{F \otimes G}$, we obtain a matrix factorization

$$Z = (A + B, A + \omega B, A + \omega^2 B, \ldots, A + \omega^{d-1} B) \in \mathrm{MF}_S^d(f+g)$$

from Lemma 6.3.1.

To finish the proof, we show that $X \widehat{\otimes} Y \cong \bigoplus_{j \in \mathbb{Z}_d} T^j(Z)$. Set $H = F \otimes_S G$ and define endomorphisms of H^d: $\tilde{A} = \operatorname{diag}(A, A, \ldots, A)$, $\tilde{B} = \operatorname{diag}(B, B, \ldots, B)$,

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1_H \\ 1_H & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1_H & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1_H & 0 \end{pmatrix}, \text{ and } D = \begin{pmatrix} 1_H & 0 & \cdots & 0 \\ 0 & \omega 1_H & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{d-1} 1_H \end{pmatrix}.$$

Then

$$\bigoplus_{j=0}^{d-1} T^j(Z) = \left(\tilde{A} + D\tilde{B}, \tilde{A} + \omega D\tilde{B}, \ldots, \tilde{A} + \omega^{d-1} D\tilde{B}\right)$$

and, using (6.1.1),

$$X \widehat{\otimes} Y = \left(\tilde{A} + C\tilde{B}, \tilde{A} + \omega C\tilde{B}, \ldots, \tilde{A} + \omega^{d-1} C\tilde{B}\right).$$

Next we construct an isomorphism $X \widehat{\otimes} Y \to \bigoplus_{j=0}^{d-1} T^j(Z)$. Let $\alpha = \sum_{i \in \mathbb{Z}_d} D^i C^{-i}$. For any $k \in \mathbb{Z}_d$, we have that

$$\begin{aligned} \alpha(\tilde{A} + \omega^{k-1} C\tilde{B}) &= \left(\sum_{i \in \mathbb{Z}_d} D^i C^{-i}\right) \left(\tilde{A} + \omega^{k-1} C\tilde{B}\right) \\ &= \sum_{i \in \mathbb{Z}_d} \tilde{A} D^i C^{-i} + \omega^{k-1} \tilde{B} D^i C^{1-i} \end{aligned}$$

since powers of C and D commute with both $\tilde{A}$ and $\tilde{B}$. On the other hand,

$$\left(\tilde{A} + \omega^{k-1} D\tilde{B}\right) \alpha = \sum_{i \in \mathbb{Z}_d} \tilde{A} D^i C^{-i} + \omega^{k-1} \tilde{B} D^{i+1} C^{-i}.$$

Since $\sum_{i \in \mathbb{Z}_d} D^i C^{1-i} = \sum_{i \in \mathbb{Z}_d} D^{i+1} C^{-i}$, it follows that the d-tuple $(\alpha, \alpha, \ldots, \alpha)$ forms a morphism of matrix factorizations. Furthermore, α is an isomorphism with inverse $\frac{1}{d} \sum_{i \in \mathbb{Z}_d} C^{-i} D^i$. Hence, $X \widehat{\otimes} Y \cong \bigoplus_{j \in \mathbb{Z}_d} T^j(Z)$ as desired. □

We have a corollary to Proposition 6.3.2 in the setting of Section 4.1. In particular, we assume $\mathbf{k}$, ω, and μ are as in Proposition 6.3.2 and let $f \in S = \mathbf{k}[\![x_1, x_2, \ldots, x_r]\!]$. Consider the functor

$$Z\widehat{\otimes}(-) : \mathrm{MF}_S^d(f) \to \mathrm{MF}_{S[\![z]\!]}^d(f + z^d)$$

where $Z = (z, z, \ldots, z) \in \mathrm{MF}_{\mathbf{k}[\![z]\!]}^d(z^d)$ and $\widehat{\otimes}$ is taken with respect to $z_1, z_2, \ldots, z_d$ where

$$z_k = \begin{cases} \omega^{k-1} & \text{if d is odd} \\ \mu\omega^{k-1} & \text{if d is even} \end{cases}.$$

For a matrix factorization $X = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in \mathrm{MF}_S^d(f)$ and $j \in \mathbb{Z}_d$, let $\mathrm{cok}_j(X) = \mathrm{cok}\,\varphi_j$. Notice that if $Y \in \mathrm{MF}_{S[\![z]\!]}^d(f + z^d)$, then $\mathrm{cok}_j(Y)$ is an MCM $R^\sharp = S[\![z]\!]/(f + z^d)$-module for all $j \in \mathbb{Z}_d$.

Corollary 6.3.3. *For any $N \in \mathrm{MCM}(R^\sharp)$ and $k \in \mathbb{Z}_d$, there is an isomorphism of $R^\sharp$-modules* $\mathrm{cok}_k\left(Z\widehat{\otimes}N^\flat\right) \cong N^{\flat\sharp}$.

Proof. Let $\varphi : N \to N$ be the S-linear map representing multiplication by z on N. By (4.1.4) we may choose

$$N^\flat = \begin{cases} (-\varphi, -\varphi, \ldots, -\varphi) & \text{if d is odd} \\ (\mu^{d-1}\varphi, \mu^{d-1}\varphi, \ldots, \mu^{d-1}\varphi) & \text{if d is even.} \end{cases}$$

Since $TZ \cong Z$ and $TN^\flat \cong N^\flat$, Theorem 6.3.2 implies that there exists $Z_0 \in \mathrm{MF}_{S[\![z]\!]}^d(f + z^d)$ such that $Z\widehat{\otimes}N^\flat \cong \bigoplus_{j\in\mathbb{Z}_d} T^j(Z_0)$. Namely, from the proof of (6.3.2), we have that

$$Z_0 = (z1_N - \varphi, z1_N - \omega\varphi, \ldots, z1_N - \omega^{d-1}\varphi)$$

where here we are identifying $S[\![z]\!] \otimes_{S[\![z]\!]} N$ with N and $1_{S[\![z]\!]} \otimes_{S[\![z]\!]} \varphi$ with φ. Using the same idea as in [Yos90, Lemma 12.2], $\mathrm{cok}(z1_N - \omega^k\varphi)$ is a free S-module for which multiplication by z is given by $\omega^k\varphi$. Equivalently, $\mathrm{cok}(z1_N - \omega^k\varphi)$ is an MCM $R^\sharp$-module isomorphic to $(\sigma^k)^*N$. Hence, $\mathrm{cok}_k(Z\widehat{\otimes}N^\flat) \cong \bigoplus_{j\in\mathbb{Z}_d}(\sigma^j)^*N$ which is isomorphic to $N^{\flat\sharp}$ by Proposition 4.3.5. □

6.3.2 Decomposability of reduced matrix factorizations

For the rest of this chapter, we restrict our attention to reduced matrix factorizations and investigate the decomposability of $X\widehat{\otimes}Y$ in this case.

Notation 6.3.4. In addition to (6.1.1), we assume X and Y are indecomposable reduced matrix factorizations of f and g respectively. We denote the number of indecomposable summands (counted with multiplicity) in the direct sum decomposition of $X\widehat{\otimes}Y$ by $\#(X\widehat{\otimes}Y)$. Set $r = \gcd(m,n)$, where m is the size of X and n is the size of Y.

Theorem 6.3.5. *The tensor product $X\widehat{\otimes}Y$ has at most dr indecomposable summands.*

Proof. Let $Z \in \mathrm{MF}^d_S(f+g)$ be a summand of $X\widehat{\otimes}Y$. By Lemma 6.2.4(i), Z_y is a summand of $\bigoplus_{j=1}^d (T^{j-1}X)^m$. Since $T^{j-1}X$ is indecomposable for each $j \in \mathbb{Z}_d$, the KRS property of $\mathrm{MF}^d_{S_1}(f)$ implies that $Z_y \cong \bigoplus_{j=1}^d (T^{j-1}X)^{r_j}$ for some integers $0 \le r_j \le m$. Hence, the size of Z_y, which is the same as the size of Z, is $(r_1 + \cdots + r_d)n$. Similarly, reduction to $S_1 = S/(x)$ gives us that the size of Z is $(s_1 + \cdots + s_d)m$ for some integers $0 \le s_j \le n$. Thus, $(r_1 + \cdots + r_d)n = (s_1 + \cdots + s_d)m$ which must be at least $\mathrm{lcm}(n,m) = nm/r$. Since $X\widehat{\otimes}Y$ is of size dnm and we have just shown that any summand must be of size at least nm/r, it follows that $X\widehat{\otimes}Y$ can have at most dr indecomposable summands. □

Theorem 6.3.6. *Suppose $X \ncong T^iX$ and $Y \ncong T^jY$ for all $i,j \neq 0 \in \mathbb{Z}_d$. Then $\#(X\widehat{\otimes}Y) \le r$.*

The proof of Theorem 6.3.6 will require some preparation. First we recall the definition and some basic properties of the *radical of an additive category*. Let $\mathcal{C}$ be a Krull-Schmidt category, that is, an additive category such that every object decomposes into a finite sum of objects which have local endomorphism rings. We refer the reader to [Kra14] for details.

For objects $X, Y \in \mathcal{C}$, let

$$\operatorname{rad}_{\mathcal{C}}(X,Y) = \{h \in \operatorname{Hom}_{\mathcal{C}}(X,Y) : 1_X - gh \text{ is invertible for all } g \in \operatorname{Hom}_{\mathcal{C}}(Y,X)\}.$$

In particular, the Jacobson radical of $\operatorname{End}_{\mathcal{C}}(X)$ coincides with $\operatorname{rad}_{\mathcal{C}}(X,X)$. We recall two useful facts about $\operatorname{rad}_{\mathcal{C}}$. Note that, while working with objects in the abstract category $\mathcal{C}$, our indices are not taken modulo d.

Lemma 6.3.7. *Let X and Y be objects in $\mathcal{C}$.*

(i) Suppose $X = X_1^{n_1} \oplus \cdots \oplus X_t^{n_t}$ and $Y = Y_1^{m_1} \oplus \cdots \oplus Y_s^{m_s}$ for indecomposable objects X_i, Y_j, and positive integers n_i, m_j, for $1 \leq i \leq t, 1 \leq j \leq s$. Then $\operatorname{rad}(X,Y) \cong \bigoplus_{i,j} \operatorname{rad}(X_i^{n_i}, Y_j^{m_j})$ for each pair i, j.

(ii) Suppose that X and Y are indecomposable objects such that $X \ncong Y$. Then $\operatorname{rad}(X^n, Y^m) = \operatorname{Hom}(X^n, Y^m)$ for any $n, m \geq 1$. □

Lemma 6.3.8. *Let $X_1, \ldots, X_m$ be indecomposable objects in $\mathcal{C}$ and $X = X_1^{n_1} \oplus \cdots \oplus X_m^{n_m}$ for positive integers $n_1, \ldots, n_m$. Let $e = (e(i,j))_{i,j}$ be an idempotent in $\operatorname{End}(X)$, where $e(i,j) : X_j^{n_j} \to X_i^{n_i}$. If $e(i,k)e(k,j) \in \operatorname{rad}(X_j^{n_j}, X_i^{n_i})$ for all i, j, k where either $i \neq k$ or $j \neq k$, then there exist idempotents $e_i \in \operatorname{End}(X_i^{n_i})$, $1 \leq i \leq m$, such that $e(i,i) - e_i \in \operatorname{rad}\operatorname{End}(X_i^{n_i})$ and $e(X) \cong \bigoplus_{i=1}^m e_i(X_i^{n_i})$, where $e(X)$ denotes the direct summand of X given by the idempotent e.*

Proof. Since $e = e^2$, for each $1 \leq i \leq m$, we have that

$$e(i,i) = e^2(i,i) = \sum_{k=1}^{m} e(i,k)e(k,i).$$

Since $e(i,k)e(k,i) \in \operatorname{rad}\operatorname{End}(X_i^{n_i})$ for all $i \neq k$ by assumption, we have that

$$e(i,i)^2 \equiv e(i,i) \mod \operatorname{rad}\operatorname{End}(X_i^{n_i}).$$

Since $\mathcal{C}$ is a Krull-Schmidt category, we may lift $e(i,i)$ to an idempotent $\operatorname{End}(X_i^{n_i})$, that is, there exists an idempotent $e_i \in \operatorname{End}(X_i^{n_i})$ such that $e(i,i) - e_i \in \operatorname{rad}(X_i^{n_i}, X_i^{n_i})$. Let $\epsilon = e_1 \oplus e_2 \cdots \oplus e_m \in \operatorname{End}(X)$ and $\gamma = e - \epsilon$. Then we claim that $\gamma^2 \in \operatorname{rad}\operatorname{End}(X)$. To see this, first notice that for each i,

$$\gamma^2(i,i) = \sum_{k \neq i} e(i,k)e(k,i) + (e(i,i) - e_i)^2 \in \operatorname{rad}(X_i^{n_i}, X_i^{n_i}).$$

Next, if $i \neq j$, then

$$\gamma^2(i,j) = (e(i,i) - e_i)e(i,j) + e(i,j)(e(j,j) - e_j) + \sum_{k \neq i, k \neq j} e(i,k)e(k,j).$$

Since $\operatorname{rad}_{\mathcal{C}}$ is an ideal in the category $\mathcal{C}$, the fact that $e(i,i) - e_i \in \operatorname{rad}(X_i^{n_i}, X_i^{n_i})$ implies that $(e(i,i) - e_i)e(i,j) \in \operatorname{rad}(X_j^{n_j}, X_i^{n_i})$. Similarly, we have that $e(i,j)(e(j,j) - e_j) \in \operatorname{rad}(X_j^{n_j}, X_i^{n_i})$ as well. Since the rest of the terms are in $\operatorname{rad}(X_j^{n_j}, X_i^{n_i})$ by assumption, we have that $\gamma^2(i,j) \in \operatorname{rad}(X_j^{n_j}, X_i^{n_i})$ for all i, j. Lemma 6.3.7(i) then implies that $\gamma^2 \in \operatorname{rad}\operatorname{End}(X)$ as claimed.

We can now finish the proof of the lemma. Since $\gamma^2 \in \operatorname{rad}\operatorname{End}(X)$, we have that $1_X - \gamma^2 = (1_X + \gamma)(1_X - \gamma)$ is a unit in $\operatorname{End}(X)$. Hence, both $(1_X - \gamma)$ and $(1_X + \gamma)$ are units in $\operatorname{End}(X)$. Note that since e and ϵ are idempotents and $\gamma = e - \epsilon$, we have the following:

$$(1_X + \gamma)\epsilon = \epsilon^2 + \gamma\epsilon = (\epsilon + \gamma)\epsilon = e\epsilon = e(e - \gamma) = e - e\gamma = e(1_X - \gamma).$$

That is, $e = (1_X + \gamma)\epsilon(1_X - \gamma)^{-1}$. It follows that $e(X) \cong \epsilon(X) = \bigoplus_{i=1}^m e_i(X_i^{n_i})$. □

If, in addition, the indecomposable objects $X_1, X_2, \ldots, X_m$ are pairwise non-isomorphic, then Lemma 6.3.7(ii) implies that the assumptions of the lemma are automatically satisfied. With this, we are able to prove (6.3.6).

Proof of Theorem 6.3.6. Suppose $e = (\epsilon_1, \ldots, \epsilon_d) \in \operatorname{End}(X \widehat{\otimes} Y)$ is an idempotent. The reduction of e to $S_2 = S/(y)$ gives us an idempotent $e_y \in \operatorname{End}((X \widehat{\otimes} Y)_y) = \operatorname{End}(\bigoplus_{q=1}^{d} (T^{q-1}X)^m)$. By Lemma 6.2.6, the pq component of e_y is the morphism

$$(\epsilon_1(p,q), \epsilon_2(p,q), \ldots, \epsilon_d(p,q))_y : (T^{q-1}X)^m \to (T^{p-1}X)^m.$$

By assumption, the indecomposable matrix factorizations $X, TX, T^2X, \ldots, T^{d-1}X$ are pairwise non-isomorphic. Hence, Lemma 6.3.8 implies that there exists idempotent endomorphisms e_i of $(T^{i-1}X)^m$, $i \in \mathbb{Z}_d$, such that

$$e_y(i,i) - e_i = (\epsilon_1(i,i), \epsilon_2(i,i), \ldots, \epsilon_d(i,i))_y - e_i \in \operatorname{rad} \operatorname{End}((T^{i-1}X)^m))$$

and $e_y((X \widehat{\otimes} Y)_y) \cong \bigoplus_{i=1}^{d} e_i((T^{i-1}X)^m)$. Since T^jX is indecomposable for all $j \in \mathbb{Z}_d$, $\bigoplus_{i=1}^{d} e_i((T^{i-1}X)^m) \cong \bigoplus_{i=1}^{d} (T^{i-1}X)^{r_i}$ for some $0 \le r_i \le m$. By Lemma A.1.3, we may pick bases so that

$$e_i = \begin{pmatrix} 1_{r_i} & 0 \\ 0 & 0 \end{pmatrix} : (T^{i-1}X)^{r_i} \oplus (T^{i-1}X)^{m-r_i} \to (T^{i-1}X)^{r_i} \oplus (T^{i-1}X)^{m-r_i},$$

where 1_{r_i} denotes the identity morphism $(T^{i-1}X)^{r_i} \to (T^{i-1}X)^{r_i}$. Since $e_y(i,i) - e_i \in \operatorname{rad} \operatorname{End}((T^{i-1}X)^m)$, Lemma A.1.2 allows us to write

$$e_y(i,i) = \begin{pmatrix} 1_{r_i} & 0 \\ 0 & A_i \end{pmatrix}$$

for some $A_i \in \operatorname{rad} \operatorname{End}((T^{i-1}X)^{m-r_i})$. Then, again applying Lemma A.1.2, we have e_y as endomorphism of $\left(\bigoplus_{i=1}^{d} (T^{i-1}X)^{r_i}\right) \bigoplus \left(\bigoplus_{i=1}^{d} (T^{i-1}X)^{m-r_i}\right)$ described as

$$e_y = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

for $A \in \operatorname{End}\left(\bigoplus_{i=1}^{d}(T^{i-1}X)^{m-r_i}\right)$, where now 1 denotes the identity morphism on $\bigoplus_{i=1}^{d}(T^{i-1}X)^{r_i}$. Therefore, we have that $e_y((X\widehat{\otimes}Y)_y) \cong \bigoplus_{i=1}^{d}(T^{i-1}X)^{r_i} \oplus \operatorname{Image} A$. This implies that $A = 0$ since $e_y((X\widehat{\otimes}Y)_y) \cong \bigoplus_{i=1}^{d}(T^{i-1}X)^{r_i}$. The diagonal components of A are $A_1, \ldots, A_d$, so $A = 0$ implies that $A_i = 0$ for all i as well. Thus, $e_y(i,i) = e_i = \begin{pmatrix} 1_{r_i} & 0 \\ 0 & 0 \end{pmatrix}$ for each $i \in \mathbb{Z}_d$. Recalling that $e_y(i,i) = (\epsilon_1(i,i)_y, \ldots, \epsilon_d(i,i)_y)$, we have that $\epsilon_k(i,i)_y$ is the identity on the first r_i summands of $(F_{k+i-1} \otimes_S G_{2-i})_y \cong F_{k+i-1}^m$ and zero on the rest. It follows that

$$\operatorname{rank}(\epsilon_k(i,i) \otimes_S \mathbf{k}) = nr_i \text{ for each } k \in \mathbb{Z}_d. \tag{6.3.1}$$

Next we consider the reduction of e to $S_1 = S/(x)$ and follow nearly identical steps. Since $Y, TY, \ldots, T^{d-1}Y$ are pairwise non-isomorphic indecomposable matrix factorizations, we can apply Lemma 6.3.8 to find idempotents $e_i' \in \operatorname{End}((T^{1-i}Y)^n)$, for $i \in \mathbb{Z}_d$, such that $e_x(i,i) - e_i' \in \operatorname{rad}\operatorname{End}((T^{1-i}Y)^n)$ and $e_x((X\widehat{\otimes}Y)_x) \cong \bigoplus_{i=1}^{d} e_i'((T^{1-i}Y)^n)$. By picking bases as above, we may write e_x as an endomorphism of $\left(\bigoplus_{i=1}^{d}(T^{1-i}Y)^{s_i}\right) \bigoplus \left(\bigoplus_{i=1}^{d}(T^{1-i}Y)^{n-s_i}\right)$, for some integers $0 \le s_i \le n$, to conclude that $e_x(i,i) = e_i' = \begin{pmatrix} 1_{s_i} & 0 \\ 0 & 0 \end{pmatrix}$ for each i. Now, since

$$e_x(i,i) = (\epsilon_1(i,i), \epsilon_2(i-1,i-1), \ldots, \epsilon_d(i+1,i+1))_x : (T^{1-i}Y)^n \to (T^{1-i}Y)^n,$$

we have that $\epsilon_k(i-k+1, i-k+1)_x$ is the identity on the first s_i summands of $(F_{i+1} \otimes_S G_{1-i+k})_x \cong G_{1-i+k}^n$ and zero on the rest. It then follows that

$$\operatorname{rank}(\epsilon_k(i-k+1, i-k+1) \otimes_S \mathbf{k}) = ms_i \text{ for each } k \in \mathbb{Z}_d. \tag{6.3.2}$$

Taking $k = 1$, we combine (6.3.1) and (6.3.2) to conclude that $nr_i = ms_i$ for all $i \in \mathbb{Z}_d$. Taking $k = 2$, we find that $ms_i = \operatorname{rank}(\epsilon_2(i-1,i-1) \otimes_S \mathbf{k}) = nr_{i-1}$ for all $i \in \mathbb{Z}_d$. Thus, $ms_1 = nr_1 = ms_2 = nr_2 = \cdots = nr_{d-1} = ms_d = nr_d$. In particular, $r_1 = r_2 = \cdots = r_d$

and $s_1 = s_2 = \cdots = s_d$ which implies that the size of the matrix factorization $e(X\widehat{\otimes}Y)$ is $\sum_{i=1}^{d} nr_i = dnr_1$ which also equals $\sum_{i=1}^{d} ms_i = dms_1$. It follows that the size of $e(X\widehat{\otimes}Y)$ is at least $\operatorname{lcm}(dn, dm) = \dfrac{d^2nm}{\gcd(dn, dm)} = \dfrac{d^2nm}{d\gcd(n, m)} = \dfrac{dnm}{r}$. The matrix factorization $X\widehat{\otimes}Y$ has size dnm and we have just shown that any summand has to have size at least dnm/r. So, $\#(X\widehat{\otimes}Y) \le dnm/(dnm/r) = r$ as desired. □

The next two results give specific situations in which $X\widehat{\otimes}Y$ is indecomposable.

Proposition 6.3.9. *Assume $X \cong TX$ and $Y = (u_1, \ldots, u_d)$ is of size 1 with $u_1, \ldots, u_d$ pairwise relatively prime elements in the maximal ideal of S_2. Then $X\widehat{\otimes}Y$ is indecomposable.*

Proof. Let $i \neq j \in \mathbb{Z}_d$. Given a commutative diagram of the form

$$\begin{array}{ccc} S_2^n & \xrightarrow{u_i} & S_2^n \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ S_2^n & \xrightarrow{u_j} & S_2^n, \end{array}$$

we have that $\alpha \otimes_{S_2} \mathbf{k} = 0 = \beta \otimes_{S_2} \mathbf{k}$. To see this, assume $\alpha \otimes_{S_2} \mathbf{k} \neq 0$. Then any matrix representing α has at least one unit entry. Since $\alpha u_i = u_j \beta$, this would imply that $u_i \in u_j S_2$, contradicting the assumption that u_i and u_j have no common factors. The same holds for β. As a consequence, any morphism $(\alpha_1, \ldots, \alpha_d) : (T^iY)^n \to (T^jY)^n$, with $i \neq j$, has $\alpha_k \otimes_{S_2} \mathbf{k} = 0$ for all $k \in \mathbb{Z}_d$.

Set $Z = X\widehat{\otimes}Y$ and let $e = (\epsilon_1, \ldots, \epsilon_d)$ be an idempotent in $\operatorname{End}(Z)$. We want to show that $e = 0$ or $e = 1$. First we consider the reduction of e to $S_2 = S/(x)$. We have that e_x is an idempotent on $Z_x \cong \bigoplus_{q=1}^{d}(T^{1-q}Y)^n$. Let $i, j, k \in \mathbb{Z}_d$ with $i \neq j$. From Lemma 6.2.7 we have that the $(i+k-1)(j+k-1)$ component of e_x is the morphism

$$(\epsilon_1(i+k-1, j+k-1), \ldots, \epsilon_k(i,j), \ldots, \epsilon_d(i+k, j+k))_x : (T^{2-j-k}Y)^n \to (T^{2-i-k}Y)^n.$$

Since $i \neq j$, we have that $\epsilon_k(i,j)_x \otimes_{S_2} \mathbf{k} = 0$ by the initial observation. It follows that

$$\epsilon_k(i,j) \otimes_S \mathbf{k} = 0 \text{ for all } i,j,k \in \mathbb{Z}_d \text{ with } i \neq j. \tag{6.3.3}$$

Next, we consider the reduction of e to $S_1 = S/(y)$ which gives us an idempotent e_y on $Z_y \cong \bigoplus_{q=1}^{d} T^{q-1}X$ since Y is of size 1. Since $X \cong TX$, each shift of X is isomorphic to X and therefore $Z_y \cong X^d$. Since X is indecomposable, the ring $\Lambda \coloneqq \operatorname{End}(X) \cong \operatorname{End}(TX) \cong \cdots \cong \operatorname{End}(T^{d-1}X)$ is local. Recall that for any $i,j \in \mathbb{Z}_d$, Lemma 6.2.6 gives us that

$$e_y(i,j) = (\epsilon_1(i,j), \epsilon_2(i,j), \ldots, \epsilon_d(i,j))_y.$$

If $i \neq j$, then (6.3.3) tells us that each of $\epsilon_1(i,j)_y, \ldots, \epsilon_d(i,j)_y$ are non-isomorphisms of free S_1-modules. This implies that the morphism $e_y(i,j)$ is a non-isomorphism of matrix factorizations. That is, $e_y(i,j) \in \operatorname{rad}\Lambda$ for all $i \neq j$, since Λ is a local ring. Hence, $e_y(i,k)e_y(k,j) \in \operatorname{rad}\Lambda$ for all triples i,j,k such that $k \neq i$ or $k \neq j$. By Lemma 6.3.8, the diagonal components $e_y(i,i)$ of e_y will give idempotents in $\Lambda/\mathrm{rad}\Lambda$, which must be 0 or 1 in the quotient. In other words, since Λ is local, Lemma 6.3.8 tells us that each $e_y(i,i)$ is either an automorphism of X or an element of $\operatorname{rad}\Lambda$.

First, assume $e_y(i,i) \in \operatorname{rad}\Lambda$ for all $i \in \mathbb{Z}_d$. Combined with the previous paragraph, this implies that all the components of e_y are in $\operatorname{rad}\Lambda$ and therefore $e_y \in \operatorname{rad}\operatorname{End}(Z_y)$ by Lemma 6.3.7(i). An idempotent in the radical of $\operatorname{End}(Z_y)$ must be zero and so we have that $e_y = 0$. Since $e_y = (\epsilon_1, \ldots, \epsilon_d)_y$, we have that $(\epsilon_k)_y = 0$ for each $k \in \mathbb{Z}_d$. In particular, $\epsilon_k \otimes_S \mathbf{k} = 0$ for each $k \in \mathbb{Z}_d$. Again, this implies that the idempotent ϵ_k must be 0 else the isomorphism $1 - \epsilon_k$ would have a non-trivial kernel. Thus, $e = 0$ in this case.

Next, assume that at least one of the diagonal components is an automorphism, that is, assume $e_y(i_0, i_0)$ is an automorphism of $T^{i_0-1}X \cong X$ for some $i_0 \in \mathbb{Z}_d$. Lemma 6.2.6 tells us that $\epsilon_k(i_0,i_0)_y$ is an isomorphism of S_1-modules for each $k \in \mathbb{Z}_d$. Since $\epsilon_k(i_0,i_0)$ is an isomorphism mod y, Nakayama's Lemma implies that it must be an isomorphism. Hence, we have that

$$\epsilon_k(i_0, i_0) \text{ is and isomorphism for all } k \in \mathbb{Z}_d. \tag{6.3.4}$$

To finish the proof, it suffices to show that $\epsilon_k(j, j)$ is an isomorphism for each $k, j \in \mathbb{Z}_d$ since then, combined with (6.3.3), we will have that each of the components $\epsilon_1, \ldots, \epsilon_d$ of e are isomorphisms. The idempotent $e = (\epsilon_1, \ldots, \epsilon_d)$ will therefore be the identity as claimed.

Let $k, j \in \mathbb{Z}_d$. In order to prove that $\epsilon_k(j, j)$ is an isomorphism, we consider reduction of e to $S_2 = S/(x)$. In particular, consider the morphism

$$e_x(k + j - 1, k + j - 1) : (T^{2-k-j}Y)^n \to (T^{2-k-j}Y)^n.$$

Since $T^{2-k-j}Y$ is a matrix factorization of size 1, any endomorphism $(\beta_1, \ldots, \beta_d)$ of $T^{2-k-j}Y)^n$ has the property that $\beta_i = \beta_j$ for all $i, j \in \mathbb{Z}_d$. By Lemma 6.2.7,

$$e_x(k+j-1, k+j-1) = (\epsilon_1(k+j-1, k+j-1), \epsilon_2(k+j-2, k+j-2), \ldots, \epsilon_d(k+j, k+j))_x.$$

Thus, we have that $\epsilon_k(j, j)_x = \epsilon_t(k+j-t, k+j-t)_x$ for all $t \in \mathbb{Z}_d$. Taking $t = k+j-i_0$, we have that $\epsilon_k(j, j)_x = \epsilon_{k+j-i_0}(i_0, i_0)_x$. By (6.3.4) and Nakayama's Lemma, we conclude that $\epsilon_k(j, j)$ is an isomorphism completing the proof. □

Proposition 6.3.10. *Suppose $Y = (u_1, \ldots, u_d)$ is of size 1. If $X \ncong T^jX$ for all $j \neq 0 \in \mathbb{Z}_d$, then $X \widehat{\otimes} Y$ is indecomposable.*

Proof. Let $e = (\epsilon_1, \ldots, \epsilon_d)$ be an idempotent in $\operatorname{End}(Z)$ where $Z = X \widehat{\otimes} Y$. We want to show $e = 0$ or $e = 1$. By Lemma 6.2.4(i), reduction of e to $S_1 = S/(y)$ gives us an idempotent

$$e_y : \bigoplus_{q=1}^{d} T^{q-1}X \to \bigoplus_{q=1}^{d} T^{q-1}X.$$

Since $X, TX, T^2X, \ldots, T^{d-1}X$ are pairwise non-isomorphic indecomposable matrix factorizations, $\operatorname{rad}(T^iX, T^jX) = \operatorname{Hom}(T^iX, T^jX)$ for all $i \neq j$. Thus, $e_y(i, k)e_y(k, j) \in \operatorname{rad}(T^{j-1}X, T^{i-1}X)$

for all i, j, k such that $i \neq k$ or $k \neq j$. We can therefore apply Lemma 6.3.8 to e_y to find idempotents $e_1, \ldots, e_d$ with $e_i \in \operatorname{End}(T^{i-1}X)$ such that $e_y(i,i) - e_i \in \operatorname{rad}\operatorname{End}(T^{i-1}X)$ and $e_y(Z_y) \cong \bigoplus_{i=1}^{d} e_i(T^{i-1}X)$. Since $T^{i-1}X$ is indecomposable, the idempotent e_i is either 0 or the identity on $T^{i-1}X$. As in the proof of Theorem 6.3.9, we consider two cases.

First, assume $e_i = 0$ for all $i \in \mathbb{Z}_d$. Then $e_y(i,i) \in \operatorname{rad}\operatorname{End}(T^{i-1}X)$ for all i. This implies that $e_y(i,j) \in \operatorname{rad}(T^{j-1}X, T^{i-1}X)$ for all $i, j \in \mathbb{Z}_d$ which, by Lemma 6.3.7(i), tells us that $e_y \in \operatorname{rad}\operatorname{End}(Z_y)$. The idempotent e_y must therefore be 0 and, again proceeding as in the proof of 6.3.9, we find that $e = 0$ in this case.

Next, assume e_{i_0} is the identity on $T^{i_0-1}X$ for some $i_0 \in \mathbb{Z}_d$. Since $e_y(i_0, i_0) - e_{i_0} \in \operatorname{rad}\operatorname{End}(T^{i_0-1}X)$, we have that $e_y(i_0, i_0)$ is an automorphism of the matrix factorization $T^{i_0-1}X$. Since

$$e_y(i_0, i_0) = (\epsilon_1(i_0, i_0), \ldots, \epsilon_d(i_0, i_0))_y$$

by Lemma 6.2.6, we have that $\epsilon_k(i_0, i_0)_y$ is an isomorphism for all $k \in \mathbb{Z}_d$. By Nakayama's Lemma, the same is true of $\epsilon_k(i_0, i_0)$, and hence $\epsilon_k(i_0, i_0)_x$, for all $k \in \mathbb{Z}_d$.

Let $j, k \in \mathbb{Z}_d$. We claim that $\epsilon_k(j,j)$ is an isomorphism. Since we already know that $\epsilon_k(i_0, i_0)$ is an isomorphism, assume $j \neq i_0$. Consider the endomorphism of $T^{2-k-j}Y$

$$e_x(k+j-1, k+j-1) = (\epsilon_1(k+j-1, k+j-1), \epsilon_2(k+j-2, k+j-2), \ldots, \epsilon_d(k+j, k+j))_x.$$

Since $T^{2-k-j}Y$ is of size 1, we have that $\epsilon_k(j,j)_x = \epsilon_t(k+j-t, k+j-t)_x$ for each $t \in \mathbb{Z}_d$. Taking $t = k + j - i_0$, we find that $\epsilon_k(j,j)_x = \epsilon_{k+j-i_0}(i_0, i_0)_x$ which is an isomorphism. Another application of Nakayama's Lemma shows that $\epsilon_k(j,j)$ is an isomorphism which completes the proof of the claim.

Since $\epsilon_k(j,j)$ is an isomorphism for all $k, j \in \mathbb{Z}_d$, it follows that $e_y(j,j)$ is an automorphism of $T^{j-1}X$ for each $j \in \mathbb{Z}_d$. Since $e_y(i,j) \in \operatorname{rad}\operatorname{End}(Z_y)$ for all $i \neq j$, we have that e_y is the sum of an automorphism of Z_y and an element of $\operatorname{rad}\operatorname{End}(Z_y)$ implying that $e_y = (\epsilon_1, \ldots, \epsilon_d)_y$ itself is an automorphism. Finally, $(\epsilon_k)_y$ being an isomorphism implies that ϵ_k

is an isomorphism for each $k \in \mathbb{Z}_d$ and hence the idempotent e must be the identity. □

Corollary 6.3.11. *Assume $d = p$ is prime and let $g = y_1 y_2 \cdots y_p \in S_2 = \mathbf{k}[\![y_1, y_2, \ldots, y_p]\!]$. For any indecomposable reduced $X \in \mathrm{MF}^p_{S_1}(f)$, the tensor product $X \widehat{\otimes} (y_1, y_2, \ldots, y_p)$ is indecomposable.*

Proof. Since p is prime, either $X \cong TX$ or $X \ncong TX$ by Lemma 4.5.1. Applying Theorem 6.3.9 if $X \cong TX$ or Theorem 6.3.10 (or Theorem 6.3.6) if $X \ncong TX$, we have that $X \widehat{\otimes} (y_1, y_2, \ldots, y_d)$ is indecomposable. □

Using Theorem 6.3.10, we obtain an extension of the results in [Yos98, §3] for the case $d = 2$.

Corollary 6.3.12. *Let $d = 2$ and assume that at least one of X or Y is of size 1. Then $X \widehat{\otimes} Y$ is decomposable if and only if $X \cong TX$ and $Y \cong TY$.*

Proof. In the case $d = 2$, $X \widehat{\otimes} Y \cong Y \widehat{\otimes} X$ [Yos98, Lemma 2.1]. So, we may assume Y is of size 1. If $X \cong TX$ and $Y \cong TY$, then $X \widehat{\otimes} Y$ decomposes by [Yos98, Lemma 3.2] or by Proposition 6.3.2 above.

Conversely, if $Y \ncong TY$, then [Yos98, Theorem 3.7] implies that $X \widehat{\otimes} Y$ is indecomposable. On the other hand, if $Y \cong TY$ but $X \ncong TX$, then Theorem 6.3.10 implies that $X \widehat{\otimes} Y$ is indecomposable. □

A | Appendix

A.1 Idempotents

Let S be a regular local ring and $d \geq 2$. Fix a non-zero non-unit $f \in S$. With the additional assumption that S is complete, Section 3.1 implies that $\mathrm{MF}_S^d(f)$ is a Krull-Schmidt category and therefore, idempotents split in $\mathrm{MF}_S^d(f)$. In this section, we give a normal form for idempotents in $\mathrm{MF}_S^d(f)$ without the assumption of completeness on S. In particular, idempotents still split in this case.

Definition A.1.1. Let $\alpha \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(X, X')$ and $\beta \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(Y, Y')$. Then the morphisms α and β are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\delta} \\ Y & \xrightarrow{\beta} & Y' \end{array}$$

where $\gamma \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(X, Y)$ and $\delta \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(X', Y')$ are isomorphisms.

When applying invertible row and column operations to the matrices in matrix factorization the options are limited in the following sense: Any invertible row operation applied to φ_k must be met with the inverse column operation on φ_{k-1} if the d-tuple is to remain a matrix factorization. The idea of the next lemma is to show that there are no restriction when it comes to morphisms, that is, we may perform any invertible row or column operations on the components of a morphism and obtain an equivalent morphism.

Lemma A.1.2. *Let $X, X' \in \mathrm{MF}_S^d(f)$ and suppose $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathrm{Hom}_{\mathrm{MF}_S^d(f)}(X, X')$. For any $k \in \mathbb{Z}_d$, replacing α_k with $P\alpha_k Q$, for invertible matrices P, Q of appropriate sizes,* *results in a morphism equivalent to α.*

Proof. Let $X = (\varphi_1 : F_2 \to F_1, \ldots, \varphi_d : F_1 \to F_d)$ be of size n, $X' = (\varphi'_1 : F'_2 \to F'_1, \ldots, \varphi'_d : F'_1 \to F'_d)$ be of size m, and let $k \in \mathbb{Z}_d$. Let $P : F'_k \to S^m$ and $Q : S^n \to F_k$ be S-isomorphisms. Then we have a commutative diagram

$$\begin{array}{ccccc} F_{k+1} & \xrightarrow{Q^{-1}\varphi_k} & S^n & \xrightarrow{\varphi_{k-1}Q} & F_{k-1} \\ \downarrow{\scriptstyle \alpha_{k+1}} & & \downarrow{\scriptstyle P\alpha_k Q} & & \downarrow{\scriptstyle \alpha_{k-1}} \\ F'_{k+1} & \xrightarrow{P\varphi'_k} & S^m & \xrightarrow{\varphi'_{k-1}P^{-1}} & F'_{k-1}. \end{array}$$

Set $Y = (\varphi_1, \ldots, \varphi_{k-1}Q, Q^{-1}\varphi_k, \ldots, \varphi_d)$ and $Y' = (\varphi_1, \ldots, \varphi'_{k-1}P^{-1}, P\varphi'_k, \ldots, \varphi_d)$. Clearly, $Y, Y' \in \mathrm{MF}^d_S(f)$. Furthermore, the commutative diagram above shows that

$$\beta = (\alpha_1, \alpha_2, \ldots, P\alpha_k Q, \ldots, \alpha_d) : Y \to Y'$$

is a morphism of matrix factorizations.

Set $\gamma = (1_{F_1}, \ldots, 1_{F_{k+1}}, Q^{-1}, 1_{F_{k-1}}, \ldots, 1_{F_d})$ and $\delta = (1_{F'_1}, \ldots, 1_{F'_{k+1}}, P, 1_{F'_{k-1}}, \ldots, 1'_{F_d})$. Then $\gamma \in \mathrm{Hom}_{\mathrm{MF}^d_S(f)}(X, Y)$, $\gamma \in \mathrm{Hom}_{\mathrm{MF}^d_S(f)}(X', Y$”, and we have that $\delta\alpha = \beta\gamma$. Since γ and δ are both isomorphisms we conclude that α and β are equivalent. □

The next lemma provides a normal form for idempotents in $\mathrm{MF}^d_S(f)$.

Lemma A.1.3. *Let $X \in \mathrm{MF}^d_S(f)$ be of size n and $e = (e_1, \ldots, e_d) \in \mathrm{End}_{\mathrm{MF}^d_S(f)}(X)$ be an idempotent. Then there exists an integer $0 \le r \le n$ such that e is equivalent to a morphism $(\epsilon, \epsilon, \ldots, \epsilon)$ where*

$$\epsilon = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} : S^r \oplus S^{n-r} \to S^r \oplus S^{n-r}.$$

In particular, there exists X' of size r and X'' of size $n-r$ such that $X \cong X' \oplus X''$.

Proof. If $e = 0$ or $e = 1$, there is nothing to prove so assume $e \neq 0, 1$. Let $k \in \mathbb{Z}_d$. Since e is an idempotent, $e_k^2 = e_k$, that is, the S-homomorphism $e_k : F_k \to F_k$ is an idempotent.

Therefore, there exists invertible homomorphisms P_k, Q_k such that

$$P_k e_k Q_k = \begin{pmatrix} I_{r_k} & 0 \\ 0 & 0 \end{pmatrix} : S^{r_k} \oplus S^{n-r_k} \to S^{r_k} \oplus S^{n-r_k}$$

for some $0 < r_k < n$. Applying Lemma A.1.2 for all $k \in \mathbb{Z}_d$ we may assume that

$$e = \left(\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} I_{r_d} & 0 \\ 0 & 0 \end{pmatrix} \right)$$

for integers $0 < r_1, r_2, \dots, r_d < n$, and that $\varphi_k : S^{r_{k+1}} \oplus S^{n-r_{k+1}} \to S^{r_k} \oplus S^{n-r_k}$.

To finish the proof, it suffices to show that $r_i = r_{i+1}$ for each $i \in \mathbb{Z}_d$. So, let $i \in \mathbb{Z}_d$ and consider the commutative diagram

$$\begin{array}{ccccc} S^{r_i} \oplus S^{n-r_i} & \xrightarrow{\varphi_{i+1}\varphi_{i+2}\cdots\varphi_{i-1}} & S^{r_{i+1}} \oplus S^{n-r_{i+1}} & \xrightarrow{\varphi_i} & S^{r_i} \oplus S^{n-r_i} \\ \downarrow{\scriptstyle e_i} & & \downarrow{\scriptstyle e_{i+1}} & & \downarrow{\scriptstyle e_i} \\ S^{r_i} \oplus S^{n-r_i} & \xrightarrow{\varphi_{i+1}\varphi_{i+2}\cdots\varphi_{i-1}} & S^{r_{i+1}} \oplus S^{n-r_{i+1}} & \xrightarrow{\varphi_i} & S^{r_i} \oplus S^{n-r_i}. \end{array}$$

Decompose φ_i and $\varphi_{i+1}\varphi_{i+2}\cdots\varphi_{i-1}$ along these direct sum decompositions into

$$\varphi_i = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \varphi_{i+1}\varphi_{i+2}\cdots\varphi_{i-1} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

Since $(\varphi_i, \varphi_{i+1}\varphi_{i+2}\cdots\varphi_{i-1}) \in \mathrm{MF}_S^2(f)$, we have that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} f \cdot I_{r_i} & 0 \\ 0 & f \cdot I_{n-r_i} \end{pmatrix}$ and $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} f \cdot I_{r_{i+1}} & 0 \\ 0 & f \cdot I_{n-r_{i+1}} \end{pmatrix}$. Since $e_i\varphi_i = \varphi_i e_{i+1}$ we have that $B = 0$ and $C = 0$. Similarly, the commutativity of the left hand square above implies that $B' = 0$ and $C' = 0$. This implies that $AA' = f \cdot I_{r_i}$ and $A'A = f \cdot I_{r_{i+1}}$ which is only possible if $r_i = r_{i+1}$ (cf. [Eis80, Corollary 5.4]).

The final statement follows by decomposing φ_i along the direct sum decomposition $\varphi_i : S^r \oplus S^{n-r} \to S^r \oplus S^{n-r}$ for all i where r is the common value $r = r_1 = r_2 = \cdots = r_d$. $\square$

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Tim Tribone | Curriculum Vitae

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Appointments

University of Utah — **Salt Lake City, UT**
Research Associate — *2022-2025*

Education

Syracuse University — **Syracuse, NY**
Ph.D. in Mathematics — *Expected Graduation: May 2022*
Advisor: Graham J. Leuschke
Thesis: *Matrix factorizations with more than two factors*

Syracuse University — **Syracuse, NY**
M.S. in Mathematics — *May 2018*

University of Rochester — **Rochester, NY**
B.S. in Mathematics — *May 2016*

Research

Research Interests

Commutative Algebra and Representation Theory.

Papers

- Graham J. Leuschke and Tim Tribone. *Branched covers and matrix factorizations.* arXiv:2110.02435. Submitted. (2021)
- Tim Tribone. *Matrix factorizations with more than two factors.* arXiv:2102.06819. Submitted. (2021)

Talks

Special Session on Commutative Algebra — **JMM**
Branched covers and matrix factorizations — *April 2022*

Commutative and Homological Market Presentations — **CHAMP**
Branched covers and matrix factorizations — *February 2022*

Route 81 Conference — **Cornell University**
Branched covers and matrix factorizations — *November 2021*

Algebra Seminar — **Syracuse University**
Matrix factorizations with more than two factors — *October 2021*

AMS Fall Central Sectional Meeting — **Creighton University**
Matrix factorizations with more than two factors — *October 2021*

UCGEN — **IMBM**
Matrix factorizations with more than two factors — *April 2021*

Mathematics Graduate Organization Colloquium — **Syracuse University**
The classification of simple hypersurface singularities — *September 2021*

Mathematics Graduate Organization Colloquium — **Syracuse University**
Factoring polynomials with matrices — *March 2021*

Graduate Conference in Algebra and Topology — **Binghamton University**
Matrix factorizations — *November 2020*

Conferences Attended

- Joint Mathematics Meeting, Seattle WA, 2022
- AMS Fall Western Sectional Meeting, (virtual) University of New Mexico, 2021
- AMS Fall Central Sectional Meeting, (virtual) Creighton University, 2021
- AMS Spring Eastern Sectional Meeting, (virtual) Brown University, 2021
- Joint Mathematics Meeting, (virtual) 2021
- Binghamton University Graduate Conference in Algebra and Topology, (virtual) Binghamton University, 2020
- AMS Fall Central Sectional Meeting, (virtual) University of Texas at El Paso, 2020
- AMS Fall Eastern Sectional Meeting, Binghamton University, 2019
- Route 81 Conference in Commutative Algebra and Algebraic Geometry, Queen's University, 2019, 2021
- Annual New York State Regional Graduate Mathematics Conference, Syracuse University, annually

Awards, Grants, and Fellowships

AMS Travel Grant - JMM
American Mathematical Society — *2022*

All-University Doctoral Prize
Syracuse University, Graduate School — *2022*
- University-wide award for superior achievement in completed dissertations

Kibbey Prize
Syracuse University, Mathematics Department — *2022*
- Departmental award for oustanding achievement in the PhD program

Outstanding Teaching Assistant Award
Syracuse University, Graduate School — *2022*
- University-wide award presented to 4% of teaching assistants annually

Graduate School Fellow
Syracuse University, Graduate School — *2016-2017, 2019-2020*

NSF Grant Support
Syracuse University, Mathematics Department — *2018*
- Received summer funding support from Professor Tadeusz Iwaniec's NSF research grant.

Teaching Experience

My teaching evaluations are available here. The missing semesters are due to the fact that I was a graduate fellow during the 2016-2017 and 2019-2020 academic years.

Instructor of Record

MAT 295: Calculus I
Syracuse University — *Fall 2021*

MAT 296: Calculus II
Syracuse University — *Spring 2021, Fall 2020*

MAT 284: Business Calculus
Syracuse University — *Summer 2020*

MAT 194: Precalculus
Syracuse University — *Spring 2022, Summer 2019, Fall 2018*

Graduate Teaching Assistant

MAT 284: Business Calculus
Syracuse University — *Spring 2019*

MAT 296: Calculus II
Syracuse University — *Spring 2018*

MAT 183: Elements of Modern Mathematics
Syracuse University — *Fall 2017*

Service/Outreach

- **AWM Treasurer**
  Association for Women in Mathematics, Syracuse University Chapter — *Fall 2020, Spring 2021*
  Co-organized career development events for graduate students including
  - A career panel of six female mathematicians in varying fields such as academia, private industry, and government.
  - A career preparation seminar led by faculty in the mathematics department.
- **Directed Reading Program Mentor**
  Syracuse University, Department of Mathematics — *Spring 2020, Fall 2020*
  Mentored two undergraduate students in semester-long independent reading projects
  - Assigned weekly reading and exercises
  - Supervised preparation of final presentations
- **Directed Reading Program Co-organizer**
  Syracuse University, Department of Mathematics — *Fall 2021, Spring 2022*
  - Redefined the standards for both undergraduate and graduate participation with the intention of increasing overall participation in the program.
  - Matched graduate students and undergraduate students based on the areas of expertise of the graduate students and the areas of interest of the undergraduates.
- **First Year TA Mentor**
  Syracuse University, Department of Mathematics — *Summer 2019*
  - I was appointed by the department to help first year graduate students prepare for their preliminary exam in Algebra.

Professional Memberships

- Association for Women in Mathematics
- American Mathematical Society
- Mathematical Association of America