Examples in Analysis

MATH 3210 – Fall 2022

University of Utah
Fall 2023

Contents

1 | The Real Numbers

1.1 Set operations and functions

Example 1.1.1. **(London Adams)** I believe that the value of this proof over the one given in the homework solutions is that we derive the formula for the number of subsets of a set rather than guessing at its form and proving it by induction. This derivation also utilizes the binomial formula and provides insight into its relation to counting n choose k things. Finally, it still uses induction to prove this formula, and so it still has the benefits and insights provided by the original proof.

Problem:

If A is a set containing n elements, how many different subsets of A can be formed? Find a formula which counts the number of subsets of A in terms of n. Keep in mind that the empty set is counted as a subset of A. For extra credit, prove your formula works.

Solution. For a set A with n elements we wish to count the number of subsets. For simplicity let's number the elements 1 through n. So $A = [1, 2, 3, 4 \cdots n-1, n]$ Each element alone is a subset, and therefor there are at least n subsets. To each of these subsets of size 1, we could take the union with any of the other (n-1) (subsets of size one. This gives us $\frac{n\,(n-1)}{2}$ subsets of size 2, where the 1/2 avoids double counting pairs of elements, since order doesn't matter. Similarly, the number of subsets of size 3 will be the number of ways to choose 3 things from a set of n things. Therefor, in general, the number of subsets of size k of a set of size n is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and the total number of subsets will just be the sum of this term over all possible sizes of subsets: Number of subsets of A is: $\sum_{k=0}^{n} \binom{n}{k}$, for $|A| = n$.

Recall the binomial formula: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. Choosing $x = y = 1$ we find that $2^n = \sum_{k=0}^{n} \binom{n}{k}$. So, the number of subsets of A is 2^n for $|A| = n$.

□

1.2 Integers and Rational Numbers

Example 1.2.1. **(Mick Wagner)**

Definition 1.2.2. A triple of integers (a, b, c) is called a *Pythagorean Triple* if

$$a^2 + b^2 = c^2.$$

Theorem 1.2.3. *Given a Pythagorean Triple, $a,\ b,\ c \in \mathbb{Z}$, there exists the following integers $p,\ q$ that the following equations hold.*

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2} \tag{1.2.1}$$

$$\frac{b}{c} = \frac{2pq}{p^2 + q^2}. \tag{1.2.2}$$

To do this, we will start with the Pythagorean Theorem. Then we will apply the quadratic formula. From there, we will look at the connections between the Pythagorean Theorem and the unit circle. Ultimately, we will explain why Equations (1.2.1) and (1.2.2) has .

Recall the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \tag{1.2.3}$$

Rearranging, observe the following.

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

Let $x = \frac{a}{c}$ and $y = \frac{b}{c} = t(x + 1)$.

$$x^2 + y^2 = 1 \tag{1.2.4}$$

$$x^2 + (t(x + 1)^2 = 1$$

$$x^2 + t^2(x + 1)^2 = 1$$

$$x^2(1 + t^2) + x(2t^2) + (t^2 - 1) = 0 \tag{1.2.5}$$

Recall that $x^2 + y^2 = 1$ is the equation of a circle with radius 1 unit, centered at the origin. Also recall the quadratic formula, shown below.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.2.6}$$

From Equation (1.2.5), observe the following quantities:

$$a_q = 1 + t^2 \qquad b_q = 2t^2 \qquad c_q = t^2 - 1 \tag{1.2.7}$$

Substitute the quantities from Equation (1.2.7) into Equation (1.2.6). This shows the following.

$$x = \frac{-2t^2 + \sqrt{(2t^2)^2 - 4(1+t^2)(t^2-1)}}{2(1+t^2)}$$

$$x_1 = \frac{-t^2+1}{t^2+1} \qquad x_2 = -1 \tag{1.2.8}$$

Note that x_2 is a trivial solution and will disregarded.

Substitute $x = x_1$ into $y = t(x+1)$.

$$y = t\left(\frac{-t^2+1}{t^2+1} + 1\right)$$

$$y = \frac{2t}{1+t^2}$$

$$\implies (x, y) = \left(\frac{-t^2+1}{t^2+1}, \frac{2t}{1+t^2}\right) \tag{1.2.9}$$

Since x, and y are rational, this implies that t is also rational (i.e. $t \in \mathbb{Q}$). Consequentially, we can say $t = \frac{q}{p}$ where $p,\ q \in \mathbb{Z}$.

Substitute t into $x = \frac{a}{c}$ and $y = \frac{q}{p}$.

$$x = \frac{a}{c} = \frac{-t^2+1}{t^2+1} \qquad y = \frac{b}{c} = \frac{2t}{1+t^2}$$

$$\frac{a}{c} = \left.\frac{-t^2+1}{t^2+1}\right|_{t=\frac{q}{p}} \qquad \frac{b}{c} = \left.\frac{2t}{1+t^2}\right|_{t=\frac{q}{p}}$$

$$\frac{a}{c} = \frac{1 - \frac{q^2}{p^2}}{\frac{q^2}{p^2}^2 + 1} \qquad \frac{b}{c} = \frac{\frac{2q}{p}}{\frac{p^2+q^2}{p^2}}$$

$$\frac{a}{c} = \frac{(p^2-q^2)p^2}{(p^2+q^2)p^2} \qquad \frac{b}{c} = \frac{(2qp)p}{(p^2+q^2)p}$$

$$\frac{a}{c} = \frac{p^2-q^2}{p^2+q^2} \qquad \frac{b}{c} = \frac{2qp}{p^2+q^2}$$

Thus we have shown that Equation (1.2.1) and Equation (1.2.2) logically follow from the Pythagorean Theorem. Thus we have completed the problem.

1.3 The Real Numbers

Example 1.3.1. **(Jaden Rosoff)**

Theorem 1.3.2. *$\sqrt[3]{2}$ is irrational.*

Proof. Assume $\sqrt[3]{2} = \frac{a}{b}$, with a and b sharing no common divisors. Cubing we get $2 = \frac{a^3}{b^3}$. So, $2b^3 = a^3$. Then a^3 is even because even numbers are defined as $j = 2k$. This means a is also even. If we substitute $a = 2k$ into $2b^3 = a^3$ we get $2b^3 = (2k)^3$. Now we have $b^3 = 8k^3$ so 2 divides b^3 as well as b. This means a and b share no common divisors and are both divisible by 2. This is a contradiction, so, $\sqrt[3]{2}$ is irrational. □

1.4 Supremum and Infimum

Example 1.4.1. **(Owen Koppe)**

This proof helped me begin to understand the idea behind using an $\epsilon > 0$ to prove a statement. I also liked this proof because it also used density of $\mathbb{R}$ and I appreciated seeing previous results being used to prove new statements. Lastly, this proof helped me see how to prove that the inf version of the statement.

Theorem 1.4.2. *Assume $L \in \mathbb{R}$ is an upper bound for a non-empty set $A \subset \mathbb{R}$. Prove that, $L = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ such that $a > L - \epsilon$.*

Example 1.4.3. Let $A = (0, 1)$, so we know that $\sup A = 1$. Let $L = 1$. Take $\epsilon = 0.001$. Consider $L - \epsilon = 1 - 0.001 = 0.999$. Now consider $a_1 \in A$ where $a_1 = 0.9999$. We can see that $a_1 > L - \epsilon$.

Proof. Let $L \in \mathbb{R}$ such that L is an upper bound for $A \subset \mathbb{R}$ where A is not the empty set. So $L \geq a$ for all $a \in A$. Let $\sup A = k$. We will show that $L = k$ if and only if for every $\epsilon > 0$ there exists $a \in A$ such that $a > L - \epsilon$.
($\Rightarrow$) Suppose that $L = k$, consider $\epsilon > 0$ such that for any $a \in A$, $a \leq L - \epsilon$. So $L - \epsilon$ is an upper bound for A. But since $L - \epsilon < k$ this contradicts $k = \sup A$ being the minimum upper bound. So for every $\epsilon > 0$ there exists some $a \in A$ such that $a > L - \epsilon$.
($\Leftarrow$) Suppose for all $\epsilon > 0$ there exists $a \in A$ such that $a > L - \epsilon$. Now suppose that $L \neq k$. Since $k = \sup A$ we know that $k < L$. By the density of $\mathbb{R}$ we know that there exists $c \in \mathbb{R}$ such that $k < c < L$. Take $\epsilon = L - c > 0$, by hypothesis we know there exists $a \in A$ such that $a > L - \epsilon = L - (L - c) = c > k$ so $a > k$. So k is not an upper bound for A. This contradicts the definition of sup because $k = \sup A$. So $L = k$.
Since we have proven that if $L = \sup A$ there exists $a \in A$ such that $a > L - \epsilon$ and that if there exists $a \in A$ such that $a > L - \epsilon$ then $L = \sup A$ we know that $L = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ such that $a > L - \epsilon$.

□

Example 1.4.4. **(Hao Yue)**

When I was doing Homework3 Q6 I find that interesting that this question actually revealed fact, please follow me now to take a revisit on that question. I will show you how I find it.

Proposition 1.4.5. *Let* $a < b$ *be real numbers and consider the set* $T = \mathbb{Q} \cap (a, b)$*. Prove that* $\sup T = b$*.*

Firstly, from what has been given, we know that the set T is a set composed only of the rationals that are contained in the set (a, b).

$$T = \{x \in (a, b) : x \in \mathbb{Q}\}$$

We also know that b is a real number that can be either Rational or Irrational. (We don't have to consider "a" since we are only find sup). So, generally, we would have two cases here if we want to find sup for the set (a, b).

1. $b \in \mathbb{Q}$
2. $b \notin \mathbb{Q}$.

Now, let's try to prove (1) and (2) and see what we get along the way:

Proof in the case (1). **W.T.S:** There is no other Q in (a, b) that is larger than b so b is the $\sup T$.

Since b lives in $\mathbb{Q}$, then we know that $b \in T$, Suppose we have a rational number x, and for all $x \in T, a < x < b$. Assume that $x = \sup T : x < b$ (For seek of contradiction) Let $w = (x + b)/2$, where $w \in \mathbb{Q}$ and $w \in (a, b)$. Then we know $w \in T$ and $w > x$ because w is the mid-point between x and b. Thus, we have $x \neq \sup T$. Contradiction by saying when $x < b$ then $x \neq \sup T$. So since $x \in T \implies x \leq b$ and by contradiction $x \nless b$, Then x can only be equal to b, $x = b$. Hence, $x = \sup T = b$. Proved. □

Proof in the case (2). **W.T.S:** There will always exist at least a rational which smaller than b that makes $b = \sup T$.

Since b doesn't live in $\mathbb{Q}$, then $b \notin T$. Assume $y = \sup T : y \in \mathbb{Q} < b$, Because $\mathbb{Q}$ is dense, so by Properties of $\mathbb{Q}$: $a \in \mathbb{R}$, $b \in \mathbb{R}$ s.t. $a < b$ then there exists $c \in \mathbb{Q}$ with $y < c < b$. We have that there exists $x \in \mathbb{Q}$ such that $y < x < b$. Thus, $x \in T$, and which implies $y \neq \sup T$. Contradiction. So by contradiction, we know that $y \nless b$ which implies $y = b$ Hence, $y = \sup T = b$. Proved. □

Example 1.4.6. **(Isaac Hodson)**

This example resonated with me because it helped me understand supremums, which is something I had struggled with previously. It also helped me understand why using arbitrary elements is useful when writing proofs. While first learning proofs, I had a hard time understanding why an element being arbitrary allowed for certain assumptions, but this example helped clear things up. In this example, when we say that M is an arbitrary upper bound, it means that M can be anything as long as it is an upper bound. This was one of the first proofs I did where I felt like I could see a clear path from what we had to what we wanted to prove. I guess in a way it helped me understand the proof writing process better, which is something I've used with pretty much every proof I've done since.

Theorem 1.4.7. $\sup(A + B) = \sup A + \sup B$

Proof. We first want to show that $\sup(A + B) \leq \sup A + \sup B$: Since $a \leq \sup A$ and $b \leq \sup B$ for any $a \in A$ and $b \in B$, $a+b \leq \sup A+\sup B$. We know that $\sup A \leq x \iff a \leq x$ for all $a \in A$, so $\sup(A + B) \leq \sup A + \sup B$.

Next we want to show that $\sup A + \sup B \leq \sup(A + B)$: Let M be an arbitrary upper bound for A + B, and fix an element $y \in A$. Since M is an upper bound of A + B, $y+b \leq M$ for all $b \in B$. $b \leq M - y$, for all $b \in B$. This means that $\sup B \leq M - y \rightarrow \sup B + y \leq M \rightarrow y \leq M - \sup B$. Since y was arbitrary, $M - \sup B \geq a$, for all $a \in A$. This tells us that $\sup A \leq M - \sup B \rightarrow \sup A + \sup B \leq M \rightarrow \sup A + \sup B \leq \sup(A + B)$.
We have proven both sides of the inequality, so therefore, $\sup A + \sup B \leq \sup(A + B)$. □

Example 1.4.8. (Garrett McClellan)

Some examples of thought experiments that have helped me grasp the concepts of the different sets of numbers

1. Can you get an irrational number from $a, b \ \in \mathbb{Q}$?

   No, think of $a \ = \ \frac{m}{n}$ *and* $b \ = \ \frac{s}{t}$ *where* $m, n, s, t \ \in \mathbb{N}$ Any form of addition, subtraction, multiplication or division will only ever give you an integer of another integer which is the definition of $\mathbb{Q}$

2. Can you get a rational number from $a, b \ \in (\mathbb{R} \ \setminus \mathbb{Q})$?

   Yes, $\sqrt{2} * \sqrt{2} \ = \ 2$ which is a rational number

3. Can you get an irrational number from $a, b \ \in (\mathbb{R} \ \setminus \mathbb{Q})$?

   Yes, $\pi * \pi \ = \ \pi^2$ which is an irrational number

4. Can you get an irrational number from $a \ \in (\mathbb{R} \ \setminus \mathbb{Q})$ and $b \ \in \mathbb{Q}$?

   Yes, $2.2 * \pi \ = \ 2.2\pi$ which is still an irrational number

5. Can you get a rational number from $a \ \in (\mathbb{R} \ \setminus \mathbb{Q})$ and $b \ \in \mathbb{Q}$?

   Yes, $\pi * 0 \ = \ 0$ which is a rational number

6. Can you get a rational number from $a \ \in (\mathbb{R} \ \setminus \mathbb{Q})$ and $b \ \in \mathbb{N}$?

   No, any integer you multiply, divide, add or subtract to an irrational number will change the value of the number, but it will remain irrational

7. Can (a, b) have a max and min if $a, b \ \in \mathbb{R}$?

   No, since there are infinitely many points between any two points in the $\mathbb{R}$ numbers that means that you can always get closer to a or b, and thus no maximum or minimum can be reached

8. Can (a, b) have a max and min if $a, b \ \in \mathbb{Q}$?

   No for the same reason as with the rational numbers. There are infinitely many points between any two points in the $\mathbb{Q}$ numbers

9. Can $(a, b) \cap \mathbb{N}$ have a max and min if $a, b \ \in \mathbb{N}$?

   Yes, if $a \ = \ 3 \ and \ b \ = \ 6 \ then \ (a, b) \ = \ (4, 5) \ = \ 4, 5$ thus the $max \ = \ 5 \ and \ min \ = \ 4$ This is because there are not infinitely many points between values

10. An example that has helped me understand the difference in sup vs max (similar proof for inf vs min)

    Consider the interval $[a, b] \ = \ I$ where $a > b \ and \ a, b \ \in \mathbb{R}$

    Let $x \ \in I$ and $M \ \in \mathbb{R}$

    $\sup(I) \ = \ M$ where $M \geq x \ \forall \ x \ \in I$ and $M \leq$ all other bounds M of I

    $\max(I) \ = \ X$ where $X \geq x \ \forall \ x \ \in I$ and $X \ \in I$

    (a) Case 1: For I = $[a, b]$

    $\sup(I) \ = \ b$ because there is no larger value in the interval than b. $b \geq x \ \forall \ x \ \in \mathbb{R} \ and \ b \leq$ all other bounds M

    $\max(I) \ = \ b$ because b is the largest value contained within the interval, there is nothing in the interval that is larger than b. MAX PROOF done in class on 10/19

    (b) Case 2: For I = (a, b)

    $\sup(I) \ = \ b$ because there is no larger value in the interval than b. $b \geq x \ \forall \ x \ \in \mathbb{R} \ and \ b \leq$ all other bounds M

    $\max(I) \ =$ **Does Not Exist**, because b is not contained within the interval and because the interval is an element of $\mathbb{R}$ numbers it means that there are infinitely many values close to b, without ever reaching it. So because there is not a single value that is larger than all the rest in the interval there is no max. MAX PROOF done in class on 10/19

Example 1.4.9. (Aidan Wilde) The triangle inequality was a large part of the class and I just looked into it a bit more and show examples and illustrations to help gain a more concrete understanding of the inequality.

The triangle inequality comes from the most basic fact that a triangle in it's most basic form, given x, y as the legs, and z as the hypotenuse, that $z \leq x + y$.

As you can see from the figures, it will always be the case that the two legs are longer than the hypotenuse, as the two legs are not taking the most direct path.

This version of the triangle inequality is not one that we used in class. In class we utilized a similar but different form. In class we replaced z, with $|x+y|$, so our equation from class is $|x+y| \leq |x|+|y|$. When we write the equation this way, we lose the relation to the triangle, as now we only have x and y which can now be interpreted as vectors.

The new way of looking at this equation is now to see what is says about the relationship between two vectors and their relationship with absolute value.

One can imagine how this fact is true, if both x and y are positive than the absolute value bars do nothing. But if either x or y is negative, by not applying the absolute value bars until after the addition of both values, we will end up with a lesser value than if we applied them before addition. Which is what the triangle inequality tells us.

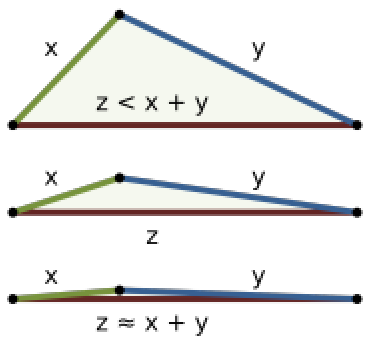


Figure 1.1: Representation of the Triangle Inequality

Example 1.4.10. (Haidong Zhao)

Let $a, b \in \mathbb{R}$ and assume $a < b$. The interval (a, b) is called an **open interval**. On the other hand, the interval $[a, b]$ is called a **closed interval**. Intervals of the form $(a, b]$ or $[a, b)$ are sometimes called "half-open" or "half-closed".

1. What is the intersection of all the open intervals containing the closed interval $[0, 1]$? Justify your answer.

   [0,1]

   Given $[0, 1] \in (a, b)$, where $a, b \in \mathbb{R}$ and assume $a < b$.

   Let $I_a = (a, b)$, $\cap_{a,b\in\mathbb{R}} I_a = \cap_{a,b\in\mathbb{R}}(a, b) = [0, 1]$

   First, let $x \in [0, 1]$. Let (a, b) be an arbitrary open set containing $[0, 1]$

   Since $x \in [0, 1] \subset (a, b)$ and $x \in [0, 1]$

   $\Rightarrow x \in (a, b)$

   Since (a, b) has arbitrary, then $\cap_{[0,1]\in(a,b)}(a, b)$

   $\Rightarrow [0, 1] \subset \cap_{[0,1]\in(a,b)}(a, b)$

   Then, let $x \in \cap_{[0,1]\in(a,b)}(a, b)$

   Suppose $x \notin [0, 1]$, then $x > 1$ or $x < 0$.

   Suppose $x > 1 \Rightarrow x - 1 > 0 \Rightarrow (-1, 1 + \frac{x-1}{2})$, but this interval still contains $[0, 1]$.

   Suppose $x < 0 \Rightarrow (-1, 1 + \frac{x-1}{2})$ is still containing $[0, 1]$.

   So, this is a contradiction because we assumes that $x \in \cap_{[0,1]\in(a,b)}(a, b)$.

   Therefore, $[0, 1]$ is the intersection of all the open intervals containing the closed interval $[0, 1]$.

2. What is the intersection of all closed intervals containing the open interval $(0, 1)$? Justify your answer.

[0,1]

Given that $(0,1) \in [a,b]$, where $a, b \in \mathbb{R}$ and assume $a < b$.

Let $I_a = [a,b]$, $\cap_{a\in\mathbb{R}} I_a = \cap_{a\in\mathbb{R}}[a,b] = [0,1]$

Example 1.4.11. **(Kristen Pimentel)**

Let $x \in \mathbb{R}$ then there exits an $n \in \mathbb{N}$ such that $x < n$ Proof: Let x be a real number. Suppose $n \leq x$ for $n \in \mathbb{N}$ then by the completeness property $\mathbb{N} \subset \mathbb{R}$ that is non-empty and bounded above there is a least upper bound b for $\mathbb{N}$ in which b is an upper bound but $b - 1$ is not. This implies there is an $n \in \mathbb{N}$ such that $b - 1 < n \implies b < n + 1$. Since b is an upper bound this is a contradiction of $\mathbb{N}$ being bounded above. Therefore every x in $\mathbb{R}$ must be less than some natural number.

Since there is a real number less than some natural number this can have multiple applications. For me, it is often difficult to apply this property solely on the original definition. The following corollaries helped me to understand why the archimedean property was useful especially in $\epsilon - \delta$ proofs.

1. If x and y are real numbers with $x > 0$ then there exists an $n \in \mathbb{N}$ such that $nx > y$ Since $x, y \in \mathbb{R}$ and $x > 0$then $\frac{y}{x} \in \mathbb{R}$ and by the AP $\exists$ an $n \in \mathbb{N}$ such that $\frac{y}{x} < n \implies nx > y$

2. If $x \in \mathbb{R}$ and $x > 0$ then there is an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$ From (1) there exists $nx > y$ which implies that there is a y such that $1 < nx$. Since $x > 0$ and $1 < nx$ then $n > 0$ shows $0 < \frac{1}{n} < x$

3. If x and y are real numbers with $x < y$ then there is a rational number $r \in \mathbb{Q}$ such that $x < r < y$ Suppose $0 \leq x$ then by the AP there is an n such that $0 < \frac{1}{n} < y - x$ Then

$$0 < 1 < n(y - x)$$

$$1 + nx < ny$$

Because $(1 + nx) \in \mathbb{R}$ then you can use the AP again. Let another rational number m exist. Then $m < 1 + nx < ny$ and $m < 1 + nx < m + 1$ then $nx < m < ny$, dividing by n you can see that $x < \frac{m}{n} < y$. Then $x < \frac{m}{n} < y$ shows that there is a rational number between two real numbers.

I think going through these proofs solidifies the concept of the Archimedean property and makes it easier, at least for me to understand and comprehend how this helps in different proofs.

Example 1.4.12. **(Hayden Soelberg)**

Using an induction proof I will show

$$\sum_{k=1}^{n} (2k)(2k-1) = \frac{n(n+1)(4n-1)}{3}.$$

Proof. Base case $n = 1$: $(2(1))(2(1) - 1) = 2 = \frac{1(1+1)(4-1)}{3}$.

Let us assume no that the statement is true for some $n \in \mathbb{N}$. We will now prove that the statement is true for $n + 1$. I.e. we will show:

$$\sum_{k=1}^{n+1}(2k)(2k-1) = \frac{(n+1)(n+2)(4(n+1)-1)}{3}.$$

The summation exapnts to:

$$\sum_{k=1}^{n+1}(2k)(2k-1) = \sum_{k=1}^{n+1}(2k)(2k-1) + 4n^2 + 6n + 2.$$

By induction, this simplifies to

$$\begin{aligned}\frac{n(n+1)(4n-1)}{3} + \frac{12n^2+18n+6}{3} &= \frac{n(n+1)(4n-1)}{3} + \frac{(12n+6)(n+1)}{3} \\ &= \frac{(n+1)(n(4n-1)+12n+6)}{3} \\ &= \frac{(n+1)(n+2)(4n+3)}{3}.\end{aligned}$$

Therefore the statement is true for all $n \in \mathbb{N}$.

□

Example 1.4.13. **(Anthony Dipasquo)**

Question: Will $\bigcap_{n=1}^{\infty}[a_n, \infty) = \emptyset$ assuming that $[a_n, \infty) \subsetneq [a_{n+1}, \infty)$.

Answer: Yes, it will. As a reminder, $[a_n, \infty) \subsetneq [a_{n+1}, \infty)$ implies that $a_n < a_{n+1}$ since a_n is on the endpoint.

Proof. Let $x \in \bigcap_{n=1}^{\infty}[a_n, \infty)$. By the Archimedian Principle, there exists $N \in \mathbb{N}$ such that $x < N$. Since $a_n < a_{n+1}$ for all $n \in \mathbb{N}$, there exist $N' \in \mathbb{N}$ such that $x < N < a_{N'}$. Hence, $x \notin [a_{N'}, \infty)$, a contradiction. □

This example shows that the Nested Interval Property is only valid for bounded intervals.

Example 1.4.14. **Subeen Lim** If $x \in \mathbb{R} \setminus \mathbb{Q}$ and $r \neq 0 \in \mathbb{Q}$, then $r + x \in \mathbb{R} \setminus \mathbb{Q}$ and $rx \in \mathbb{R} \setminus \mathbb{Q}$.

This expression helps me understand what $\mathbb{R}$ is and what $\mathbb{Q}$ is. Also, teaching about properties of rational numbers.

Proof. Suppose $r + x \in \mathbb{Q}$. The sum of two rational numbers is a rational number. So, if $r + x \in \mathbb{Q}$ and $r \in \mathbb{Q}$, then $r + x - r = x \in \mathbb{Q}$, contradiction.

Similarly, suppose $rx \in \mathbb{Q}$. The division of two rational number is a rational number. So, if $rx \in \mathbb{Q}$ and $r \in \mathbb{Q}$, then $\frac{rx}{r} = x \in \mathbb{Q}$, contradiction. □

Example 1.4.15. (Tim Tribone) The following series of examples show that the intersection or union of *infinitely many* subsets of $\mathbb{R}$ can have surprising results.

1. Based on simple computations, one might suspect that the intersection of open intervals is always again an open interval. For example, if you intersect the intervals $I = (0, 3)$ and $J = (1, 4)$ you obtain $I \cap J = (1, 3)$ which is again an open interval. The following example (and its variations) show that this can fail in a several different ways for the intersection of infinitely many open intervals:

   - $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1]$
   - $\displaystyle\bigcap_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right) = (0, 1]$
   - $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1\right) = [0, 1)$

   Not only is the infinite intersection of these open intervals not necessarily an open interval, it can a closed interval or a half-open/half-closed interval of either type.

2. Similarly, the union of infinitely many closed intervals can have equally suprising results.

   - $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0, 1)$
   - $\displaystyle\bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = [0, 1)$
   - $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = (0, 1]$

Each of the six examples listed can be proved using the Archimedean Property of $\mathbb{R}$. For instance, we will provide a proof of the first intersection.

Proof. To start, notice that $[0, 1] \subset \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$ for all $n \geq 1$. Hence $[0, 1] \subset \displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$.

To show the reverse containment, suppose there exists $x \in \displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$ such that $x \notin [0, 1]$. In particular, either $x > 1$ or $x < 0$.

If $x < 0$, then by the Archimedean Property, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < -x$. It follows that $-\frac{1}{n} > x$ which implies that $x \notin \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$. This is a contradiction and so the case $x < 0$ is not possible.

If $x > 1$, then, again by the Archimedean Property, there exists $n \in \mathbb{N}$ such that $0 < x - 1 < \frac{1}{n}$. However, this implies that $x < 1 + \frac{1}{n}$ again leading to a contradiction.

Thus, no element of the intersection can live outside of $[0, 1]$. Together with the first half of the proof, we can conclude that $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1]$. □

2 | Sequences

2.1 Sequences

Example 2.1.1. **(Subeen Lim)** Prove that $\{\sqrt{n}\}_{n=1}^{\infty}$ converges to 0.

I think this is the most basic proof problem of limits. This problem tells me which N to pick and how that N is used. So it will serve as a basis for proving various other limit formulas.

Proof. Let $\varepsilon > 0$. By the A.P. there exists $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon^2} < N$. Then, for any $n \geq N$,

$$\left|\frac{1}{\sqrt{N} - 0}\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} = \varepsilon\,.$$

□

Example 2.1.2. **(Aidan Wilde)** This is an example from the worksheet we did on class on 9/16/22, which the majority of the class including me had issues with. I'd figure I'd attempt to give a somewhat thorough attempt of solving it.

The question posed was the following: Prove that $\lim_{n\to\infty} \frac{2n-1}{n+1} = 2$ As shown during lecture, the 4 recommended steps to solve problems like this are

1. Let $\epsilon > 0$ be given
2. Demonstrate a choice $N \in \mathbb{N}$ (this step usually involves A.P.)
3. Assume $n \geq N$
4. Show that your choice of N works
5. Show $|a_n - L| < \epsilon$

To begin writing our solution lets go to the first step of our procedure: Let $\epsilon > 0$ The next step now instructs us to demonstrate a choice of N, but before we do this we must do some background work to determine what inequality we will be applying to N. The background work we must do is simplify the expression $|\frac{2n-1}{n+1} - 2|$, this expression simplifies to $\frac{3}{n+1}$ Now that we have the simplified version of the original expression, we can set the inequality involving ϵ, $\frac{3}{n+1} < \epsilon$ Now we must solve this expression for n, which yields: $\frac{3}{\epsilon} - 1 < n$ Now that we have done this background work, we can define N, we will define N using the A.P., we will state by the A.P. $\exists N \in \mathbb{N}$ such that $\frac{3}{\epsilon} - 1 < N$ notice that all we have done here is substitute little n for big N, we will use the fact that little n is greater than this expression later. Also by the A.P. we assume that for any $n \geq N$ $|\frac{3}{n+1}| = \frac{3}{n+1} < \frac{3}{N+1} < \frac{3}{(\frac{3}{\epsilon}-1)+1} = \frac{3}{\frac{3}{\epsilon}} = \epsilon$

Which proves our original statement $\frac{2n+1}{n+1} - 2 < \epsilon$

Example 2.1.3. **(Anthony Dipasquo)** Not all unbounded sequences have a limit of ∞ or $-\infty$.

Consider the sequence $\{a_n\} = \{(-1)^n \cdot n\}_{n=1}^{\infty}$.

This is an alternating unbounded sequence which goes towards both infinity and negative infinity. Since the values are not going to the same value, the limit does not exist. There is also no bound on this since as n gets higher the absolute value of each value of each value in the sequences does as well.

Therefore, $\{(-1)^n \cdot n\}$ is an example of an unbounded sequence that does not have a limit of infinity or negative infinity.

This is important because it is a reminder that not all unbounded sequences have a limit of $\pm\infty$.

2.2 Monotone Convergence Theorem

Example 2.2.1. **(London Adams)** This example highlights the fact that the Monotone Convergence Theorem is not an "if and only if" statement by providing a sequence which is not monotone, but still converges.

Consider the sequence $a_n = \frac{1}{n}(-1)^n$ This sequence is clearly not monotone, since its sign flips with each subsequent term. We will now prove that it converges regardless.

Proof:
Let $\epsilon > 0$ be given
$|\frac{1}{n}(-1)^n - 0| = |\frac{1}{n}||(-1)^n|$
$\rightarrow |\frac{1}{n}||(-1)|^n = |\frac{1}{n}|1^n = |\frac{1}{n}|$
By A.P. there exists $N \in \mathbb{N}$ s.t. $0 < \frac{1}{N} < \epsilon$
Since $n > N \rightarrow \frac{1}{N} > \frac{1}{n}$
$\rightarrow |a_n - 0| = |\frac{1}{n}(-1)^n| = \frac{1}{n} < \epsilon$ Whenever $n > N$ for some $N \in \mathbb{N}$
This proves that the limit of the sequence is 0, despite it not being monotone. □

2.3 Subsequences and the Bolzano-Weierstrass Theorem

Example 2.3.1. **(Isaac Hodson)**

Theorem 2.3.2. *If $\{a_n\}$ has a limit, then each of its sub-sequences has the same limit.*

Proof. Let $\{a_{nk}\}$ be a sub-sequence of $\{a_n\}$. Therefore $\{n_k\}$ is an increasing sequence of natural numbers. This implies that $n_k \geq k$ for all k. Let $\lim a_n = L$. Let $\epsilon > 0$ be given such that there is an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for $n > N$. Therefore $k > N$ implies

$n_k > N$ since $n_k \geq k$. We can then say that $|a_{nk} - L| < \epsilon$ when $k > N$. This implies that $\lim a_{nk} = L$. □

This example from the textbook resonated with me because it helped me better understand the Bolzano-Weierstrass theorem. When we first learned about the B.W. theorem, I had a hard time understanding the concept of convergent sub-sequences, and sub-sequences in general. Going through this proof from the textbook helped me better understand those concepts because it explains why sequences with limits share that limit with all of its sub-sequences in a fairly simple and intuitive way. My understanding of this proof has helped greatly in understand proofs involving the B.W. theorem and Cauchy sequences.

Example 2.3.3. (Hao Yue & Haidong Zhao)
Goal: Prove that $\cos(n)$ has a convergent subsequence and explain why it would be hard to explicitly write down the subsequence.

First of all, notice that the original sequence $\{cos(n)\}$ is divergent but with bound $(-1\ ,\ 1)$ since it is periodic. Therefore, it is possible for us to find such convergent subsequences of it as long as the subsequence is monotone(Case 1) or has the exact same value for every element(Case 2).

Thus, there are two ways to find such a convergent subsequence out of the original sequence $\{cos(n)\}$.

Case 1:

Consider if you only pick $n(s)$ term from the original sequence with conditions such that

$$n = P_k(2\pi) + R_k,$$

$$\text{where } R_k \text{ is the remainder of } n\ mod(2\pi),\ s.t.\ R_k\ >\ R_{k+1},$$

$$P_k = n\ mod(2\pi)$$

$$\forall P,\ k\ \in \mathbb{N}\ and\ R \in\ \mathbb{R}$$

So when $n \to \infty$

1. Monotonically Increasing :

$$\{cos(n)\} \to\ 1\ (\text{local Maxs' of } \cos(n))$$

The starting point should be in the interval where the function $cos(n)$ is increasing so that we get images of our inputs by the $cos(n)$ function on the circle which are larger/higher than the previous one through jumped-picking process by conditions on the n's until reached the local Maxs' of $cos(n) = 1$ So that the Supreme of this subsequence is the same as the local Maxs' of $cos(n) = 1 \Rightarrow \sup\{a_{nk} \mid n \in \mathbb{N}\} = 1$

$$OR$$

2. Monotonically Decreasing :

$$\{cos(n)\} \to -1\ (\text{local Mins' of} \cos(n))$$

Exact same logic but opposite direction here, Your starting point should be in the interval where the function $cos(n)$ is decreasing so that you get images of your inputs by the $cos(n)$ function on the circle which are less/lower than the previous one through jumped-picking process by conditions on your n's until reached the local Mins' of $cos(n) = -1$. So that the Infimum of this subsequence is the same as the local Mins' of $cos(n) = -1 \Rightarrow \inf\{ a_{nk} \mid n \in \mathbb{N}\} = -1$

The reason that is hard to explicitly write down the subsequence because it is very hard for us to know the pattern on how k grows in the equation $n = P_k(2\pi) + R_k$, since we have $P_{k+1} - P_k \not\equiv C$ (Constant) $\forall k \in \mathbb{N}$, in other word, it's really hard to know how far should we jump to get the next n that satisfies the condition $R_k > R_{k+1}$.

Case2:

In this case we are considering to form a convergent subsequence in a way that we only pick $n(s)$ such that its image by the function $\cos(n)$ are all the same. In this way, every element in the subsequence is the same and therefore convergent.

Thus, we are picking n such that for example: $a_{nk} = \cos(nk) \equiv 1$

$\Rightarrow$ Then this subsequence would look like:

$\{a_{nk}\} = \{\cos(n_1), \cos(n_2), \cos(n_3), ..., \cos(nk)\} = \{1, 1, 1, ..., 1\}$

$\Rightarrow$ Since $a_{nk} = \cos(nk) \equiv 1$, we know that $\lim a_{nk} = \lim \cos(nk) = \lim 1 = 1$

$\Rightarrow \lim a_{nk} = 1$

$\Rightarrow$ Subsequence$\{a_{nk}\}$ converges to 1.

We found that it is nearly impossible to find a particular function for n because $n \in \mathbb{N}$, so we need jump to get our desired $n(s)$. By using "Desmos Graphing Calculator", we can find that the first integer with image 1 by function $\cos(n)$ is at 710, and yet haven't found the next integer that satisfies the condition until $n = 5000$.

x	$\cos(x)$
0	1
710	1
711	0.54025157
712	−0.41620166
713	−0.990001
714	−0.65359799
715	0.28372
4384 more rows Show all	
5100	−0.36689877
5101	0.58455128
5102	0.99856758
5103	0.49450545

Figure 2.1: Outputs of $\cos(x)$ which are near 1

x	$\cos(x)$
0	1
710	1
711	0.54025157
712	−0.41620166
713	−0.990001
714	−0.65359799
715	0.28372
4384 more rows Show all	
5100	−0.36689877
5101	0.58455128
5102	0.99856758
5103	0.49450545

Example 2.3.4. (Anthony Dipasquo)

An example that not all unbounded sequences have a limit of either $+\infty$ or $-\infty$.

$$\{(-1)^n \cdot n\}_{n=1}^{\infty}$$

This is an alternating unbounded sequence which goes towards both infinity and negative infinity. Since the values are not going to the same value, the limit does not exist. There is also no bound on this since as n gets higher so is the absolute value of each term is the sequence.

This is important because it is a reminder that not all unbounded sequences have a limit of $\pm\infty$.

2.4 Cauchy Sequences

Example 2.4.1. (Owen Koppe)

Suppose a sequence $\{a_n\}$ has the property that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_{n+1} - a_n| < \varepsilon \text{ whenever } n \geq N.$$

Is $\{a_n\}$ necessarily Cauchy? Prove it is or give a counterexample.
Counterexample: Consider the sequence:

$$0, 1, 1/2, 0, 1/4, 2/4, 3/4, 1, 7/8, 6/8, 5/8, 4/8, 3/8, 2/8, 1/4, 0, 1/16, ...$$

For every $\epsilon > 0$ we can choose $N \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ whenever $n \geq N$. Clearly this sequence is not Cauchy since it is not convergent.

Consider $\epsilon = 0.3$ then let $N = 4$ and $n = 4$ so $a_n = 0$ and $a_{n+1} = 1/4$. So $|a_{n+1} - a_n| = |0 - 1/4| = 1/4 < \epsilon < 0.3$. So our sequence satisfies the condition for an epsilon. But clearly the sequence is not Cauchy since it does not converge.

When I first approached this problem I thought it was for a true statement as it looks very similar to the definition of Cauchy. After much thinking and looking at different sequences I came across the above sequence that is very clearly not Cauchy and isn't even monotone. This sequence illustrated for me the importance of the arbitrary selection of $m, n > N$ in the definition of Cauchy. Before seeing this example I didn't think it was particularly important because I had not considered the case where the sequence is not monotone.

Example 2.4.2. (Teric Abunuwara)
When I was studying for the final I was reviewing the Bolzano-Weierstrass Theorem. When practicing the proof of the Theorem stating that all Cauchy sequences are convergent and the Nested Interval Property I found a problem that I thought was interesting and wanted to try.
Question: Start with the Bolzano-Weierstrass Theorem and use it to construct a proof of the Nested Interval Property.

Proof. Let $I_n = [a_n, b_n]$ be nested closed intervals.

By the BW Theorem, there exists a convergent subsequence of a_n, a_{n_k} such that $a_{n_k} \to a$ for some $a \in \mathbb{R}$. Since $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, we can conclude that $a = \lim a_{n_k} = \sup\{a_{n_k}\}$. It follows that $a \in I_n$ for all n. Hence the intersection is non-empty as asserted by the Nested Interval Property. □

Example 2.4.3. (Kristen Pimentel)
Problem: Show that $a_n = \frac{n^2}{n^2+1}$ is Cauchy and find what it converges to.

In order to show a sequence is Cauchy we can use the Cauchy Criterion and know that if a_n converges then it is Cauchy.

In other words a_n is Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ then $|a_n - a_m| < \varepsilon$, where the distance between two points is made arbitrarily small ultimately showing the convergence because they are approaching the same point.

Proof. Let $\varepsilon > 0$ be given. By AP there exsists $N \in NN$ such that $N > \sqrt{2/\varepsilon}$. Then, for all $n, m > N$

$$\begin{aligned}
|a_n - a_m| &= \left| \frac{n^2}{n^2+1} - \frac{m^2}{m^2+1} \right| \\
&= \left| \frac{n^2 - m^2}{(n^2+1)(m^2+1)} \right| \\
&\leq \left| \frac{n^2}{(n^2+1)(m^2+1)} \right| + \left| \frac{m^2}{(n^2+1)(m^2+1)} \right| \\
&\leq \frac{n^2}{n^2(m^2+1)} + \frac{m^2}{m^2(n^2+1)} \\
&= \frac{1}{m^2+1} + \frac{1}{n^2+1} \\
&\leq \frac{1}{m^2} + \frac{1}{n^2}.
\end{aligned}$$

Since $n, m \geq N > \sqrt{2/\varepsilon}$, we have that $\frac{1}{n^2} < \varepsilon/2$ and similarly, $\frac{1}{m^2} < \varepsilon/2$. Thus,

$$|a_n - a_m \leq \frac{1}{m^2} + \frac{1}{n^2} < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for any $n, m \geq N$ and the sequence is Cauchy and therefore converges. □

Doing this example was helpful, especially when understanding how to simplify $\frac{n^2-m^2}{(n^2+1)(m^2+1)}$ further into finding a good term for N so that we can show $|a_n - a_m| < \varepsilon$. I think that looking at $\frac{n^2}{n^2+1}$ and using L'Hosipital's rule to find

$$\lim \frac{n^2}{n^2+1} = \lim \frac{n}{n} = 1$$

and seeing it converges we can show that it's Cauchy but the more rigourous proof is good practice because finding out what N should be is not always clear.

3 | Continuous Functions

3.1 Continuous Functions

Example 3.1.1. (Mick Wagner)

I didn't have a good grasp of the differences between a "ε" proof and a "$\varepsilon - \delta$" proof. This problem on HW 7 helped me to clarify the strategy for approaching a "$\varepsilon - \delta$" proof. Intuitively I believe there exists a case, n for all $n > 0 \in \mathbb{N}$ by using a loose notation, but I am unsure how to do it rigorously. A theorem in the textbook backs my intuition, without the rigorous proof.

Let $g(x) = \sqrt[3]{x}$.

Prove that g is continuous at 0.

Proof. Take $c = 0$. Observe $|x - c| = |x - 0| = |x| < \delta$ and let $\delta = \varepsilon$. Then,

$$|g(x) - g(c)| = |g(x) - g(0)| = |g(x) - 0| = |g(x)| = |\sqrt[3]{x}| < |x| < \delta = \varepsilon.$$

Since we have shown $|g(x) - g(c)| < \varepsilon$ for $c = 0$, we know g is continuous at 0.

While this is somewhat straightforward, I learned that it is valid to let $\delta = \varepsilon$. This comes from us being able to fix the relationship between an arbitrary δ and all ε. $\square$

Prove that g is continuous at $c \neq 0$.

Hint: *Use the identity* $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Proof. For next case, take $c > 0$. Observe $|x - c| < \delta$. When $\delta = \min\{\frac{c}{2}, \varepsilon c^{\frac{2}{3}}\}$, then,

$$|g(x) - g(c)| = |\sqrt[3]{x} - \sqrt[3]{c}| = \left| \left(\sqrt[3]{x} - \sqrt[3]{c}\right) * \frac{x^{\frac{2}{3}} + \sqrt[3]{x}\sqrt[3]{c} + c^{\frac{2}{3}}}{x^{\frac{2}{3}} + \sqrt[3]{x}\sqrt[3]{c} + c^{\frac{2}{3}}} \right| = \frac{|x - c|}{\left| x^{\frac{2}{3}} + \sqrt[3]{x}\sqrt[3]{c} + c^{\frac{2}{3}} \right|}$$

$$\leq \frac{|x - c|}{c^{\frac{2}{3}}} < \frac{\delta}{c^{\frac{2}{3}}} = \varepsilon.$$

For case $c < 0$, we have that $\delta = \min\{\frac{c}{2}, \varepsilon c^{\frac{2}{3}}\}$. Then this follows from the case $c > 0$.

Then, we have shown $|g(x) - g(c)| < \varepsilon$ and consequentially g is continuous when $c \neq 0$.

This part of the problem led me to realize that there likely exists an expansion for $a^n - b^n = (a - b)$("lesser order terms") that we can use in the scratch work to determine a relevant δ for the specific n. $\square$

Example 3.1.2. **(Aidan Wilde)** While reading the textbook I ran over this theorem. The ideas about continuity made sense, but I also like seeing a visual aide to go along with the written math. That's why I created a desmos at the following link:

Desmos Link

You should be able to play around with the functions/values if you wish.

Example 3.1.3. **(Johnny Riches)**

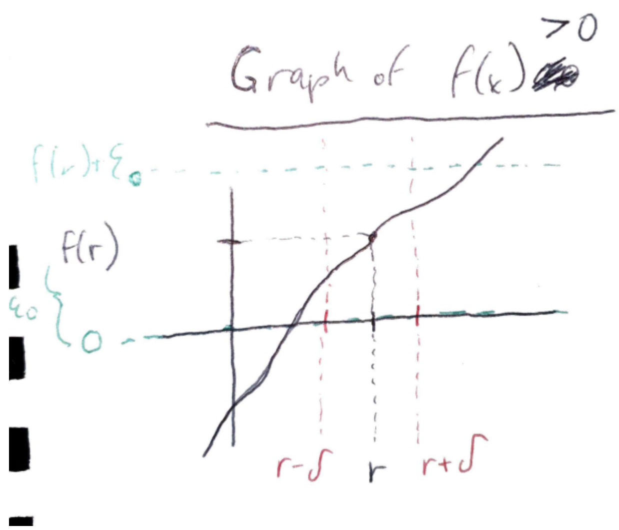
In proving the IVT, we needed to address a special case where there exists some c in $f(c)$ such that $f(c) = L$, where $L =$ some real number between $f(a)$ and $f(b)$. This special case was when $f(c) = L = 0$, and at first the conclusion of this proof confused me. Now, however, I better understand the proof of the special case, thanks to a more thorough explanation I received, alongside drawing it out on a graph.

The conclusion that we came to for this proof was a contradiction, where our proof was showing that $f(x_0) \leq 0$, yet also $f(x_0) > 0$. $f(x_0) \leq 0$ came from the condition under which we defined the set$k$, assuming $f(a) < 0 < f(b)$, where $K = \{x \in [a, b] | f(x) \leq 0\}$, and plugging x_0 into K. $f(x_0) > 0$ came from creating a value $r = \sup K$, and assuming $f(r) > 0$; we then set $\varepsilon_0 = f(r)$, which means that there exists some $\delta > 0$ such that, whenever $|x - r| < \delta$, it implies $|f(x)–f(r)| < \varepsilon_0 = f(r)$. Then, by simplifying the inequality, we find that $f(x) > 0$, which implies that $f(x_0) > 0$.

Particularly, proving that $f(x_0) > 0$ confused me at first, because I thought this was suggesting that $f(x_0) > 0$ was being compared to the value of 0, i.e. the function when using x_0 is always > 0. This thinking may not have been totally wrong, but it didn't paint the full picture for me; in reality, we arrived at this conclusion by simplifying the inequality $|f(x)–f(r)| < f(r)$, first by rewriting bounds for the absolute value as

$$-f(r) < f(x)–f(r) < f(r)$$

, then by adding $f(r)$ to all sides of the inequality to get $f(x) > 0$. My picture I drew for this problem better illustrates this, as the focus on this proof is more on the arbitrary values of ε_0 and δ in relation to $x, f(x), r$, and $f(r)$, rather than the literal numerical value of $f(x)$.



Example 3.1.4. **(Sidnee Wood)** I like this example because we use a delta epsilon proof to prove the limit of something. It's good practice for finding delta and practicing writing these types of proofs.

Prove $\lim_{x\to 3}(2x+1) = -5$.

Scratch: We want to find δ so when $|x-3| < \delta$ so that the distance between f(x) and 5 is less than ϵ, $|(-2x+1)-(-5)| < \epsilon$

$|(-2x+1)-(-5)| < \epsilon$

$|-2x+6| < \epsilon$

We want this equation to look like $|x-3| < \delta$.

$|-2(x-3)| < \epsilon$

Because of the absolute values, $|-2(x-3)| = |2(x-3)|$.

$2|(x-3)| < \epsilon$

$|(x-3)| < \frac{\epsilon}{2}$.

Now for the proof:

Let $\epsilon > 0$ be given.

choose $\delta = \frac{\epsilon}{2}$.

Assume $|(x-3)| = \frac{\epsilon}{2}$.

Now $|(-2x+1)-(-5)| = |-2x+6| = 2|x-3| = 2 \times \frac{\epsilon}{2} = \epsilon$.

Thus, $|(-2x+1)-(-5)| < \epsilon$.

Example 3.1.5. **(Jaden Rosoff)** I have had a hard time with continuity in the past and I found that this example helps me with continuity and epsilon-delta proofs.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + 2x - 1$. We will show that f is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|x-1| < \delta \implies |f(x)-f(1)| < \varepsilon$.

Notice that

$$|f(x) - f(1)| = |f(x) - 2| = |x-1||x^2 + x + 3|.$$

Let $\delta = \min\{1, \varepsilon/9\}$. Then, for any x with $|x-1| < \delta$ we have that

$$|f(x) - f(1)| = |x-1||x^2 + x + 3| \leq |x-1|(2^2 + 2 + 3) < 9\delta = 9(\varepsilon/9) = \varepsilon$$

since $|x-1| < 1 \implies 0 < x < 2$. □

Example 3.1.6. **(Adam Losser)** If a function is continuous on a partially closed interval, the result of the extreme value theorem cannot be guaranteed.

Consider the function $f(x) = \dfrac{1}{(x-4)^2}$. $f(x)$ is continuous on the interval $I = (4, 6]$. Recall that f attains a maximum on I only if there exists $s \in I$ such that $f(s) \geq f(x)$ for all $x \in I$. Similarly, f attains a minimum on I only if there exists $s \in I$ such that $f(s) \leq f(x)$ for all $x \in I$.

We see that for the specific function and interval given above, f attains a minimum on I because $f(6) \leq f(x)$ for all $x \in I = (4, 6]$. However, f does not attain a maximum on I because there is not $s \in I$ such that $f(s) \geq f(x)$ for all $x \in I$. In other words, no matter how close x gets to 4, one can always generate another x-value closer to 4 that yields a larger y value when plugged into the function f.

Therefore, functions must be continuous on a fully closed intervals to guarantee that they will attain both a max and a min.

This helped me because I tend to gloss over whether intervals are closed, open, or a combination of both. It's important to consider the behavior of the function at the endpoints of its domain to avoid any surprises.

Example 3.1.7. **(Kristen Pimentel)** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(r) = 0$ for all $r \in \mathbb{Q}$. Prove that f is the constant function 0, i.e., $f(x) = 0$ for all $x \in \mathbb{R}$.

This problem was on the 10/21 worksheet however I don't think the majority of people solved it in class.

First, I want to restate what it means to be a continuous function.

Definition 3.1.8. A function $f : D \to \mathbb{R}$ is continuous at $a \in D$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ when $x \in D$ and $|x - a| < \delta$.

The rational and irrational numbers are dense in the real numbers meaning that there is a rational number, respectively an irrational number, between any pair of distinct real numbers. I.e., for $a, b \in \mathbb{R}$ and $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Suppose $f \neq 0$, that is suppose there exists $c \in \mathbb{R} \setminus \mathbb{Q}$ such that $f(c) = y$ for $y \neq 0$. Let $n \in \mathbb{N}$. By the Intermediate Value Theorem, there exists $r_n \in \mathbb{Q}$ such that $c < r_n < c + \frac{1}{n}$ and $f(r_n) = 0$. Thus, we have a sequence of rational numbers $\{r_n\}$ converging to c. Since f is continuous, we must have that $0 = \lim f(r_n) = f(\lim r_n) = f(c) = y$, a contradiction. Hence $f(x) = 0$ for all $x \in \mathbb{R}$.

Example 3.1.9. **(Anthony Dipasquo)**

Let $f(x) = \begin{cases} x & x \neq 7 \\ 8 & x = 7 \end{cases}$. This is an example of a function which is uniformly continuous on the Domain $(-\infty, 7) \cup (7, \infty)$ but because it has one point that is not equal it could not be uniformly continuous from $(-\infty, \infty)$. So if asked if f is uniformly continuous on $(1, 10)$ the answer would be no.

This helped me really understand what uniform continuity meant, as I know that it is not just asking if a function is continuous on the given interval.

Example 3.1.10. **(Peter Chan)**

Can a continuous function be discontinuous? An interesting question that I have been asking myself is that if it is possible to have both continuity and discontinuity at the same time for a function. The definition for continuity claimed in class is enough to claim that an equation like $1/x$ is continuous. That is because the domain of function is within $\mathbb{R} \setminus \{0\}$, thus x is within $\mathbb{R} \setminus \{0\}$. If we take account 0 we can make a continuous function become discontinuous.

Example 3.1.11. **(Salar Ahmed)**

For this project submission, I decided to create a modified version of Problem 3 from Exam 2 and solve it.

Prove, using an ϵ - δ proof, that $f(x) = \sqrt{2x + 10}$ is continuous at a = 3.

Proof. Let $\epsilon > 0$ be given. Notice that

$$\mid f(x) - f(3) \mid = \mid \sqrt{2x+10} - 4 \mid = \frac{|\sqrt{2x+10}-4||\sqrt{2x+10}+4|}{|\sqrt{2x+10}+4|} = \frac{|2x+10-16|}{|\sqrt{2x+10}+4|} = \frac{|2x-6|}{|\sqrt{2x+10}+4|}.$$

Since $\sqrt{2x+10} \geq 0$, we have that $\sqrt{2x+10} + 4 \geq 4$. Therefore,

$$\frac{|2x-6|}{|\sqrt{2x+10}+4|} \leq \frac{|2x-6|}{4}$$

. Thus, if we pick $\delta = 4\epsilon$, we have that for $x \geq -10$ with $\mid 2x - 6 \mid < \delta$,

$$\mid f(x) - f(3) \mid = \frac{|2x-6|}{\sqrt{2x+10}+4} < \frac{\delta}{4} = \frac{4\epsilon}{4} = \epsilon.$$

This implies that f is continuous at $a = 3$. □

By creating my own version of a previous exam problem, I'm showing that I have a clear understanding of how to prove continuity of a function at point. The reason this example was helpful to my understanding of proving continuity is because the steps in the solution were clearly defined. This example also gives me a template that I can follow for problems similar to this one that will be on the final exam. Problems about proving limits, continuity, and derivatives have been some of my favorites ones to solve in this class because they tend to follow the same pattern of using the $\epsilon - \delta$ method for the proof.

Example 3.1.12. **(Subeen Lim)**

Theorem 3.1.13. *If $f_n : D \to \mathbb{R}$ is a sequence of continuous functions on D which converge uniformly to $f : D \to \mathbb{R}$, then f is continuous on D*

This Theorem shows two definitions (Uniform Convergence and continuity). So, I can understand easily if there are many steps how to prove. It's the perfect proof to read carefully in order to understand the relationship between uniform convergence and continuity.

4 | The Derivative

4.1 Derivative Rules

Example 4.1.1. **(London Adams)** The following is a simple derivation of the quotient rule as a special case of the product and chain rules.

This proof assumes that the product rule and chain rule have already been proven.

Proof. We wish to find $(\frac{f(x)}{g(x)})'$
$\frac{f(x)}{g(x)} = f(x)\frac{1}{g(x)}$
$\Rightarrow (\frac{f(x)}{g(x)})' = (f(x)\frac{1}{g(x)})'$

Applying the product rule
$(\frac{f(x)}{g(x)})' = f'(x)\frac{1}{g(x)} + f(x)(\frac{1}{g(x)})'$

Applying the chain rule to the second term
$\Rightarrow (\frac{f(x)}{g(x)})' = f'(x)\frac{1}{g(x)} + f(x)(-\frac{1}{g(x)^2}g'(x))$
$Rightarrow(\frac{f(x)}{g(x)})' = f'(x)\frac{1}{g(x)} - f(x)g'(x)\frac{1}{g(x)^2}$
$\Rightarrow (\frac{f(x)}{g(x)})' = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}$
$\Rightarrow (\frac{f(x)}{g(x)})' = \frac{f'(x)g(x)}{g(x)^2} - \frac{f(x)g'(x)}{g(x)^2}$
$\Rightarrow (\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ □

Example 4.1.2. **(Sidnee Wood)** I like this example because we the take limits from the left and right sides of a function and we use the definition of a derivative to show that the function is differentiable at a specific point. This problem is similar to question 2 on homework 8.

Prove that $f(x)$ is continuous and prove that it is differentiable at $x = 3$.

$$f(x) = \begin{cases} x^2+2 & x \geq 3 \\ 6x-7 & x < 3 \end{cases}$$

Proving continuity: If the function is differentiable it has to be continuous so we will prove continuity.

$f(3) = (3)^2 + 2 = 9 + 2 = 11$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 6x - 7 = 18 - 7 = 11$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 + 2 = 9 + 2 = 11.$

Then the $\lim_{x\to 3} f(x) = 11$.

Therefore the function is continuous.

Now prove that the function is differentiable at $x = 3$.

We will use the definition $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{f(3+h)-f(3)}{h} = \lim_{x\to 0^-} \frac{(6(3+h)-7)-(3^2+2)}{h} = \lim_{x\to 0^-} \frac{6h}{h} = \lim_{x\to 0^-} 6 = 6.$$

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{x\to 0^+} \frac{((3+h)^2+2-(3^2+2)}{h} \\ &= \lim_{x\to 0^+} \frac{9+6h+h^2+2-11}{h} \\ &= \lim_{x\to 0^+} \frac{6h+h^2}{h} \\ &= \lim_{x\to 0^+} 6+h \\ &= 6. \end{aligned}$$

Thus, $f'(3) = 6$. So $f(x)$ is differentiable at $x = 3$.

4.2 L'Hospital's Rule

Example 4.2.1. **(Jaden Rosoff)** I think it is important to understand how to use L'Hospital's Rule, which is something I struggled with in Calculus. Here is an example that uses L'Hospital's Rule and the Chain Rule:

Consider $\lim\limits_{x\to 0} \dfrac{\arcsin(4x)}{\arctan(5x)}$ which is a indeterminate form of type "$\frac{0}{0}$". We have that

$$\lim_{x\to 0}\frac{\arcsin(4x)}{\arctan(5x)} = \lim_{x\to 0}\frac{\dfrac{4}{\sqrt{1-(4x)^2}}}{\dfrac{5}{1+(5x)^2}}$$
$$= \lim_{x\to 0}\frac{4}{5}\frac{1+25x^2}{\sqrt{1-16x^2}}$$
$$= \frac{4}{5}\cdot\frac{1+25(0)^2}{\sqrt{1-16(0)^2}}$$
$$= \frac{4}{5}.$$

Example 4.2.2. **(Adam Losser)** Demonstration that the converse of Rolle's Theorem is not true.

Rolle's Theorem states that if a function f is continuous on the closed interval $[a,b]$ an differentiable on (a,b) such that $f(a)=f(b)$ then $f'(x)=0$ for some $x\in(a,b)$.

However, if $f'(x)=0$ for some $x\in(a,b)$ it is not necessarily true that $f(a)=f(b)$.

Example: Consider $f(x)=x^3$ defined on $[-1,1]$. The function f is continuous and differentiable on $[-1,1]$. Moreover, $f'(x)=3x^2$ so $f'(0)=0$. However, $f(1)\neq f(-1)$.

This was helpful to me because sometimes I am tempted to assume that converses of theorems are true when they're not. Unless you can prove it, never assume that the converse of a theorem is true.

4.3 The Mean Value Theorem

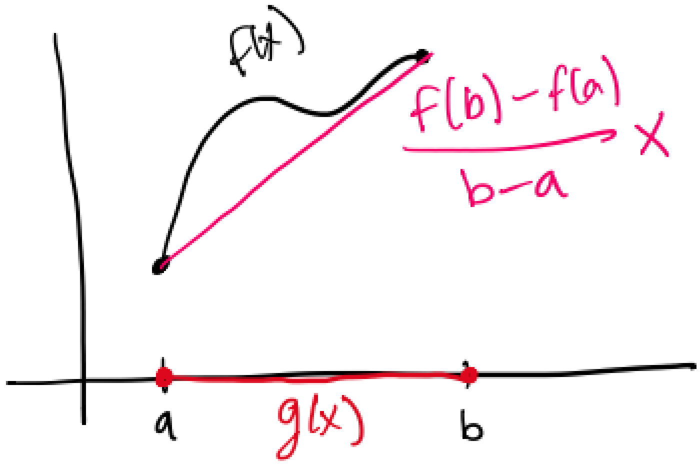
Example 4.3.1. **(Teric Abunuwara)**

When I was studying the proofs of theorems for the final I had a hard time finding a proof of the Mean Value Theorem that didn't involve using Rolle's Theorem or some other theorem from class. I did some research on stack exchange and found a proof that was compelling to me.

Proof. Consider two functions $f(x)$ and $g(x)$ defined on $[a,b]$. Let $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}x$.
Note: $\frac{f(b)-f(a)}{b-a}$ is the equation of the secant line. The first proof I found defined a new function $h(x)=f(x)-g(x)$ and had $g(a)=f(a)$ and $g(b)=f(b)$, but I found this way to be clearer.

Since g is continuous on $[a,b]$, there are two cases to consider:

Case 1: $g(x)$ attains a maximum and a minimum at b and a. The graph of this case could look like this:

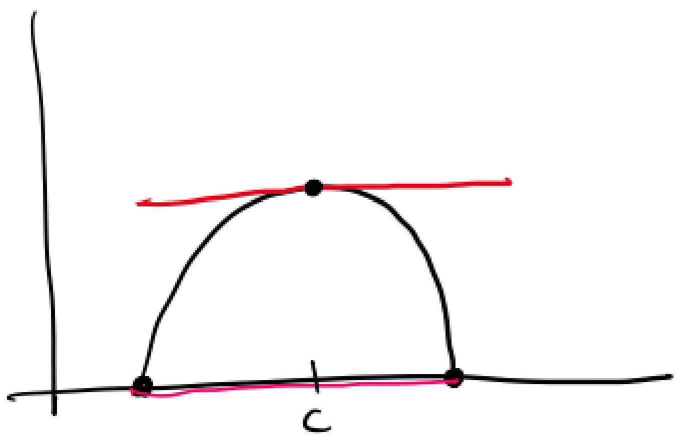


This means g is a constant. In this case, if we take the derivative we get $0 = g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. This implies $f'(x) = \frac{f(b)-f(a)}{b-a}$ for all $x \in (a, b)$. This satisfies the conclusion of the Mean Value Theorem.

Case 2: If the maximum or minimum of g is attained at an interior point $c \in (a, b)$ then $g'(c) = 0$. Therefore,

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \implies f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The graph of this case looks like this:



□

Example 4.3.2. **(Kimball Schipaanboord)**

When going over my notes, worksheets, theorems, etc. I was reading over the Practice Exam 2 handout. The last question on that handout mentions the *Fixed Point Theorem*. I remember hearing of that theorem in past courses and assumed it was in some relation to other necessary theorems for this course, i.e. Mean Value Theorem and Intermediate Value Theorem. Specifically, the question on the practice exam sheet was hinting at applying the Intermediate Value Theorem. I realized we never directly proved the Fixed Point Theorem, so as to not to do the exact problem on the exam sheet and still apply the aforementioned theorems for studying, I found a direct proof for the Fixed Point Theorem while using the Mean Value Theorem

Corollary 4.3.3. *If f is differentiable with $f'(x) > 1$ for all x, then there is at most one c such that $f(c) = c$.*

Proof. Let $g(x) = f(x) - x$ and $g'(x) = f'(x) - 1 > 0$ (have that $f'(x) > 1$ so for $f'(x) - 1$ will be grather than 0.)

If there were distinct a and b with $f(a) = a$ and $f(b) = b$ then from equation of $g(x)$, have $g(a) = g(b) = 0$ by *Mean Value Theorem* there is a c such that $g'(c) = \frac{g(b)-g(a)}{b-a}$ but $g(a) = g(b)$ so $g'(c) = 0$ this cannot be true since $g'(x) = f'(x) - 1 > 0$ for all x.

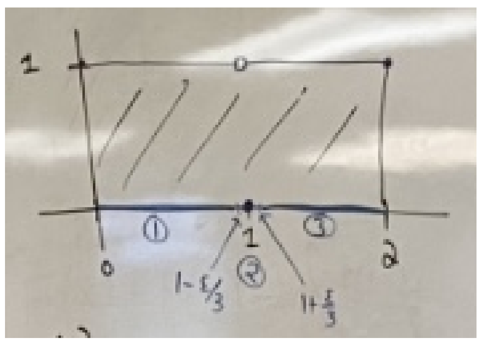
Thus there can NOT be distinct a, b such that $f(a) = a$ and $f(b) = b$, so there is at most one fixed point. □

5 | The Riemann Integral

5.1 Integration and Discontinuites

Example 5.1.1. (Salar Ahmed) One example from class that helped me to gain a good understanding of integrating a discontinuous function was the proof of how to solve $\int_0^2 f(x)dx$ where

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 1 \end{cases}$$



(Tribone)

First, let's look at how we would solve this integral using calculus, and then we'll provide a proof for our answer.

$$\int_0^2 f(x)dx = 1(2-0) = 2$$

Proof: Claim: f is integrable on [0,2].
For any partition, P, of [0, 2], U(f, P) = 2.
However $L(f,P) < 2$: Let $\epsilon > 0$ be given.
Consider the partition $P_\epsilon = \{0 < 1 - \frac{\epsilon}{3} < 1 + \frac{\epsilon}{3}\}$.
So, $L(f,P)$
$= (1 - \frac{\epsilon}{3})(1) + (\frac{2}{3}\epsilon)(0) + (1 - \frac{\epsilon}{3})(1)$
$= 2 - \frac{2}{3}\epsilon$
Now look at: $U(f,P_\epsilon) - L(f,P_\epsilon) = 2 - (2 - \frac{2}{3}\epsilon)$
$= \frac{2}{3}\epsilon < \epsilon$.
So, f is integrable. □

The reason this example from class was helpful for my understanding of integrating discontinuous functions is because the function that was used was simple and the proof that went along with this example was easy for me to understand. I think that the picture Professor Tribone drew along with the example was also very helpful!

I think another thing to mention here is that not all discontinuous functions are integrable. One example of a discontinuous function that is not integrable is the function that has a value of 1 on every rational point and a value of 0 on every irrational point. The reason this function is not integrable is because for any partition of [0,1] every subinterval will have parts of the function at height 0 and at height 1, so there' no way to make the Riemann sums converge. The only way we could integrate this type of function is with a Lebesgue integral.

Example 5.1.2. **(Owen Koppe)**

Theorem 5.1.3. *If f is a bounded function on a closed bounded interval $[a, b]$ and f is continuous except at finitely many points of $[a, b]$, then f is integrable on $[a, b]$.*

I found this theorem very counter intuitive when presented in class. I wanted to better understand the idea so I did some research and proved it. I also like the strong induction approach as it simplifies the problem a lot.

Proof. We will prove this using strong induction. We will do induction on the number of discontinuities of f on $[a, b]$.

Base Case: 1 discontinuity, let the discontinuous point be c. We know that c can either be a, b or some point in between a and b. If c is a middle point then we can break the integral into two portions, one from $[a, c]$ and one from $[c, b]$ by addition of integrals we know f is integrable on $[a, b]$. Now we will consider when c is an endpoint. We will prove this when $c = b$ ($c = a$ follows similarly). Choose $x_1 \in [a, b)$ close to b such that $(M_n - m_n)(b - x_1) < \epsilon/2$. This happens when $(M_n - m_n) < 2\sup\{f(x) : x \in [a, b]\}$. We know f is continuous on $[a, x_1]$ since there are no discontinuities on this interval so it is integrable on this interval. Therefore, there is a partition, G, of $[a, x_1]$ such that $U(G, f) - L(G, f) < \epsilon/2$. Now let $P = G \cup \{b\}$ be a partition of $[a, b]$. So:

$$U(P, f) - L(P, f) = U(G, f) - L(G, f) + (M_n - m_n)(x_1, b) < \epsilon/2 + \epsilon/2 = \epsilon$$

. Therefore, f is integrable on $[a, b]$ when there is one discontinuity.
Inductive step:
Assume f is integrable on $[a, b]$ with at most k discontinuities. We will now consider f with $n + 1$ discontinuities. Let x_k be one of these discontinuities. Break $[a, b]$ into 3 sub intervals: $[a, x_k]$, $[x_k, x_{k+1}]$, and $[x_{k+1}, b]$ where x_{k+1} is another discontinuous point in $[a, b]$. We know in $[a, x_k]$ there will be no more then k discontinuities, and in $[x_k, x_{k+1}]$ there will be no more then k discontinuities, and lastly, in $[x_{k+1}, b]$ there will be no more then k discontinuities. Therefore, by the strong induction hypothesis we know each on of these intervals is integrable. Therefore by addition of integrals we know that $[a, b]$ is integrable with $k+1$ discontinuities. Therefore, we have proven that if f is a bounded function on a closed bounded interval $[a, b]$ and f is continuous except at finitely many points of $[a, b]$, then f is integrable on $[a, b]$ □

5.2 Integration Trick and Techniques

Example 5.2.1. **(Isaac Hodson)** For my final project submission, the example is chose is the proof of U-Substitution:

Let $g : I \to \mathbb{R}$ be differentiable and assume g' is integrable on I. Let J = g(I). Then, for any $a, b \in I$, $\int_a^b f(g(x)) * g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Proof: Set $F(x) = \int_{g(a)}^{x} f(u)du$. Since $(F(g(x))' = F'(g(x)) * g'(x)$, we know that it also $= f(g(x)) * g'(x)$. So, $\int_a^b f(g(x)) * g'(x)dx = F(g(b)) - F(g(a)) = F(g(b)) = \int_{g(a)}^{g(b)} f(u)du$.

This proof at first seemed very confusing and messy, but now it has come to be one that I appreciate. Once I was able to look at it and understand where each ' was and what it meant for each function, I was able to get a better grasp on it. For example, $(F'(g(x)) * g'(x) = f(g(x)) * g'(x)$, because $F'(x)$ is the same as $f(x)$. Simplifying each step like this really helped me get a better understanding of this proof and proofwriting in general.

Example 5.2.2. **(Mick Wagner)** I was struggling to make the connection intuitively from what I could graph in Desmos to the integral on paper. This helped me clarify the strategy for understanding the behavior of improper integrals.

Prove that the improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Proof. The first step is to break apart the behavior by introducing a limit. Then the integral can be taken in the usual sense. Observe,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-p}\,dx\right) = \lim_{b\to\infty}\left(\frac{x^{-p+1}}{-p+1}\bigg|_1^b\right) = \lim_{b\to\infty}\left(\frac{b^{1-p}}{-p+1} - \frac{1}{-p+1}\right).$$

Notice, $\frac{1}{-p+1}$ cannot affect the limit. So we care about $\lim_{b\to\infty} b^{1-p}$. This is the dominating behavior of the improper integral.

Break the remaining portion into cases where p is a constant.

If $p > 1$, then $1 - p < 0$ and $\lim_{b\to\infty} b^{1-p}$ does not exist (i.e. diverges).

If $p < 1$, then $1 - p > 0$ and $\lim_{b\to\infty} b^{1-p}$ exists (i.e. converges).

If $p = 1$, then $\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}(\ln|b| - \ln|1|)$ does not exist (i.e. diverges). □

Example 5.2.3. **(Jingze Zhang)**

Find $\int_0^x t^n \ln(t)\,dt$ where n is an arbitrary integer.

Let $x > 0$, $g'(t) = t^n$ and $f(t) = \ln(t)$. Then $g(t) = \frac{1}{n+1}t^{n+1}$ and $f'(t) = \frac{1}{t}$. Both f and g are continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. So, fg' and gf' are continuous.

$$\frac{1}{n+1}t^{n+1}\ln(t)\bigg|_0^x - \int_0^x \frac{1}{n+1}t^{n+1}\frac{1}{t}\,dt = \frac{1}{n+1}x^{n+1}\ln(x) - \frac{1}{(n+1)^2}x^{n+1}$$

6 | Infinite Series

6.1 A First Look at Series

Example 6.1.1. **(Garrett McClellan)** Prove that the harmonic series $S_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges two different ways

Proof (1). $\frac{1}{2}$ proof
$S_1 = 1$, $S_2 = 1 + \frac{1}{2}$, $S_3 = 1 + \frac{1}{2} + \frac{1}{3}$, $S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
$S_4 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4})$
$S_4 > 1 + \frac{1}{2} + \frac{1}{2} \Rightarrow S_4 > \frac{4}{2}$
$S_8 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) \Rightarrow S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \rightarrow S_4 > \frac{5}{2}$
In general for $S_{2^k} = 1 + \frac{1}{2} + . . . + (\frac{1}{2^{k-1}+1} + . . . \frac{1}{2^k})$
$\Rightarrow S_{2^k} > 1 + k(\frac{1}{2})$
Notice $lim_{k\to\infty} S_{2^k} > lim_{k\to\infty}(1 + \frac{k}{2})$
Since $lim_{k\to\infty}(1 + \frac{k}{2}) \rightarrow \infty$ then $lim_{k\to\infty} S_{2^k} \rightarrow \infty$
Thus S_{2^k} diverges

□

Proof (2). Integral Test
Let $A_n = \frac{1}{n}$ and $f(x) = \frac{1}{x} \Rightarrow A_n = f(n)$
Recall that for the integral test f(x) must be positive and monotonically decreasing
For $n = [1, \infty)$, $\frac{1}{n} > 0$ thus this function is always positive
For $x, y \ \epsilon \ [1, \infty)$ *and* $x < y \Rightarrow \frac{1}{x} > \frac{1}{y}$ for any x,y in the interval thus this function is monotonically decreasing
For this test use $\displaystyle\int_1^{\infty} f(x)\, dx = \int_1^{\infty} \frac{1}{x}\, dx = lim_{n\to\infty} \int_1^{n} \frac{1}{x}\, dx$
$\rightarrow lim_{n\to\infty} log(n) - log(1) = \infty$
So by the integral test $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

□

Example 6.1.2. **(Kimball Schipaanboord)**

As we just begin our studying series in class, I wanted to freshen up on my understanding of past concepts of series and simply series as a I missed out on the majority of them in my past Calculus II course. I also remember in class hearing the mention of telescoping series and wanted to research more into those as well. This lead me across this problem that seemed able to be solved after the first class on series, contained all these parts, yet simple enough to understand for me.

Problem: Prove that the series $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and find its sum.

Proof. We can rewrite $\frac{1}{n(n+1)}$ using partial fraction decompoistion by doing this:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \implies 1 = (n+1)A + nB.$$

This implies that $A = 1$ and $B = -1$.

Now, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ so the nth partial sum of S is:

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

This is an example of a telescoping series, as a telescoping series is a series when a general term t_n that can be written as $t_n = a_n - a_{n+1}$, that is, the difference between two consecutive terms in some sequence a_n.

Then we have $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)$ gives us that S converges and $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

I found this especially interesting as in class we found that $\sum_{n=1}^{\infty} \frac{1}{n}$ the harmonic series diverges. □

6.2 Series Tests

Example 6.2.1. **(Johnny Riches)** For the number of times that we've used the ratio test for power series in recent lectures/homework, I wanted to try and fully understand why the ratio test suffices as a form of testing for a radius/interval of convergence. Intuitively, it wasn't super clear to me at first, because although the process of solving via the ratio test made sense, I didn't understand why doing $|a_{n+1}/a_n|$ would lead us to a conclusion relating to convergence. However, after looking around online, I found a good explanation for what the ratio test is really doing. Namely, the part of the ratio test that tells us that a series converges so long as $L < 1$ (where $L =$ the limit as n approaches infinity of a_{n+1}/a_n) can be used to write out an inequality where:

$a_{n+1}/a_n \leq r < 1$, for some $r < 1$, given $L < 1$, whenever $n \geq N$. From here, it's a matter of rearranging the inequality to get to $a_{N+n} \leq r^n a_N$, which more clearly displays convergence (via the comparison test). This method provides a very cut-and-dry explanation to how the ratio test works, and it's been very helpful to me in better understanding the work being done in the background of the ratio test. I've also linked a screenshot I took from the person who gave this proof online, which elaborates in a few areas that I didn't, as well as shows more scratch work for how we arrived at this conclusion.

Example 6.2.2. **(Adam Losser)** An example of two convergent series $\sum a_k$ and $\sum b_k$ such that the series $\sum a_k b_k$ diverges.

Consider the following two series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}.$$

Both series converge by the Alternating Series Test. However

$$\sum_{k=1} a_k b_k = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}k^{1/3}} = \sum_{k=1} \frac{1}{k^{5/6}}$$

diverges by the p-series test because $p = 5/6 < 1$.

In general, if we have series of the form

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/p}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/q}}$$

they will converge for any $p, q \in (0, \infty)$. However, $\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{p}+\frac{1}{q}}}$ will diverge if $\frac{1}{p} + \frac{1}{q} < 1$, equivalently, if $q + p < pq$.

Something interesting about this inequality: If $q = 1$ or $p = 1$, the inequality will always be false (e.g. if $p = 1$, we have that $q + 1 < q$ which is false).

The Point: If we have convergent series of the form

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/p}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/q}}$$

where $p, q \in (0, \infty)$, then if $p = 1$ and/or $q = 1$, then $\sum_{k=1}^{\infty} \left(\frac{(-1)^k}{k^{1/p}} \frac{(-1)^k}{k^{1/q}} \right)$ converges. But, if $p, q > 1$, the term by term product series diverges.

This was helpful to me because it shows that we can have two convergent series whose term by term product series is convergent or divergent. You can't multiply convergent series term by term and expect for the result to be convergent.

Example 6.2.3. **(Kyle Kwon)**

I found that, throughout this semester, not all the problems we did would always converge because the $\lim_{x\to\infty} a_n = 0$. There are some that diverge even with $\lim_{x\to\infty} a_n = 0$.

Theorem 6.1.2 (Term Test) If a series $a_1 + a_2 + a_3 + ... + a_k + ...$ converges, then $\lim_{x\to\infty} a_n = 0$.

Proof. Does the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)}$. converge?

Solution: Let's apply the term test, the result is

$$\lim_{x\to\infty} \frac{1}{(k+1)} = \lim_{x\to\infty} \frac{\frac{1}{k}}{(1+\frac{1}{k})} = 0.$$

The limit is 0, but the series is still divergent because this is a harmonic series. All harmonic series tend to diverge since they are monotonically decreasing. Meaning, that they decrease per term. You can also use the Limit Comparison Test on a harmonic series to show that it diverges.

Example 6.2.4. **(Kyle Kwon)** A p-series is a series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}.$, where $p > 0$. Prove that a p-series converges if and only if $p > 1$.

I wanted to see if this example is always true, so here are two cases to see if this statement holds up:

Case 1: $p > 1$

Let $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^2}$.

By using the Integral Test, we can see that this function is a positive, non-increasing function on $[1, \infty]$ using Desmos. Since the improper integral of this series converges, then the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

Case 2: $p < 1$

Let $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^{-2}}$.

By finding the limit of the series, we can see that the $\lim_{x \to \infty} \frac{1}{k^{-2}} = \infty$. According to the Series Divergence Test, a series is divergent if the limit is nonzero or does not exist.

Thus, we have shown that the p-series test holds true for both cases.

Example 6.2.5. **(Kyle Kwon)**

Let $\{a_k\}$ be a non-increasing sequence of non-negative numbers which converges to 0. Then the series

$\sum_{k=1}^{\infty} (-1)^{k+1} * a_k = a_1 - a_2 + a_3 - a_4 + ...$

converges. In fact, if s_n is the nth partial sum of this series and $s = \lim_{x \to \infty} s_n$, then

$|s - s_n| \quad \leq \quad a_{n+1}$ for all n.

I wanted to see if there is a way to do a divergent alternating series as the ones we practiced with and the ones in the book all have convergent alternating series using the following series as an example:

$\sum_{k=1}^{\infty} \frac{(-1)^k}{x^{-2}}$

Using the series above, we get our terms:

$1 - 4 + 9 - 16 + 25 - 36 + ...$

While the alternating series diverges, it's not actually a divergent alternating series. Rather, it is an alternating p-series. Alternating p-series mostly converge conditionally, as long as $0 < p \leq 1$. Thus, an alternating series must be either absolutely or conditionally convergent, depending on whether if it's a p-series or not.

Example 6.2.6. **(Jingze Zhang)**

Prove that $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin(kx)}{2^k}$ is continuous on the entire real line.

Proof. Notice that $\sin(kx)2^{-k} \leq 2^{-k}$ and $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$. By the M-test, $\sum_{k=1}^{\infty} \frac{\sin(kx)}{2^k}$ converges uniformly for all $x \in \mathbb{R}$. The uniform limit of continuous functions is continuous so this proves that the series is continuous on the entire real line. □

6.3 Power and Taylor Series

Example 6.3.1. **(Salar Ahmed)**

For this project submission, I decided to create my own problem regarding writing the power series representation for a function.

Find a power series representation for $g(x) = ln(6 - x)$.

Proof. First, notice that,

$$ln(6 - x) = \text{-}\int \tfrac{1}{6-x}\, dx$$

and then recall that we can write a power series representation for

$$\frac{1}{6-x}.$$

$$\begin{aligned}\ln(6-x) &= -\int \frac{1}{6-x}\, dx \\ &= -\int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\, dx \\ &= C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}\end{aligned}$$

We can find the constant of integration by plugging in a value for x. Let's plug in $x = 0$ since this will give us an easy computation to deal with.

$$\begin{aligned}\ln(6-0) &= C - \sum_{n=0}^{\infty} \frac{0^{n+1}}{(n+1)5^{n+1}} \\ &= C - 0 \\ \implies \ln(6) &= C\end{aligned}$$

So the power series representation is

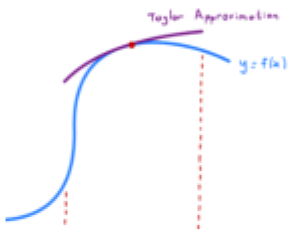
$$\ln(6-x) = \ln(6) - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}$$

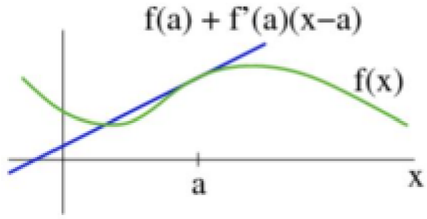
□

The reason this example was helpful to my understanding of power series representations is that it made me realize power series for natural logarithmic functions are based on a modified integral of the function that you're dealing with. If a similar problem to this one were to come up on the exam, then I can could use this same to find the power series for another natural logarithmic function. Overall, I feel that I have a good understanding of power series representations of functions.

Example 6.3.2. **(Peter Chan)**

Taylors theorem is very important in approximating functions that we can show in a Taylors polynomial. The degree of choice is up to us and the higher the degree it is more specific.





These diagrams help make better intuitions for me. By increasing the degrees of n higher we get a closer result to our desired function.

Example with Euler's formula explained Taylor's series definition. This is a very important example that allows me to remember the expansion of e^x and $\cos x$ and $\sin x$ in Taylor series. Euler's formula is e^{ix} If we simply expand the exponential, we may get the following. $1 + ix + (ix)^2/2! + \cdots$. With the $i^{\text{even numbers}}$, we get -1. Thus the infinite Taylor expansion becomes reasonable to separate the expansion into

$$(1 - x^2/2! + \cdots) + i(x - x^3/3! + \cdots)$$

Thus becomes $\cos x + i \sin x$. This is a very direct proof of a famous formula and therefore shows the powerful results that Taylor's theorem brings to mathematics.