Matrix factorizations with more than two factors

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Let S be a regular local ring.

Fix $f \in S$ (non-zero / non-unit)

$(S = \mathbb{C}[\![x_1, x_2, \ldots, x_n]\!])$

<u>Theme:</u> Compare maximal Cohen-Macaulay modules over

$$R = S/(f) \quad \text{and} \quad R^\# = \frac{S[\![z]\!]}{(f + z^d)},$$

$$d \geq 2.$$

(assume S is complete char $k \nmid d$)

Def. A finitely generated module M over a local ring A is maximal Cohen-Macaulay (MCM) if

$$\operatorname{depth} M = \dim A$$

(For Gorenstein A, MCM = Gproj)

For this talk: $S \twoheadrightarrow R$

$$M \text{ is MCM over } R \iff \operatorname{pd}_S M = 1$$

Let $MCM(A)$ = category of MCM A-modules.

Why these modules?

(1) (Buchweitz) For a Gorenstein ring A,

$\vdots$

$$\underline{\mathrm{MCM}}(A) \simeq \frac{D^b(A)}{\mathrm{Perf}(A)}$$

$\rightsquigarrow$ Homological perspective

(2) (Auslander) For an S-algebra A

MCM(A) has AR sequences $\iff$ A is an isolated singularity.

$\rightsquigarrow$ Representation theoretic perspective

$\vdots$

A local ring A has finite CM type if $MCM(A)$ has only finitely many indecomposable objects up to iso.

Q: When does a local ring have finite CM type?

Hard question.

Theorem. (Knörrer 87') $d=2$

R has finite CM type $\iff$ $R^{\#} = \dfrac{S[[z]]}{(f+z^2)}$ has finite CM type.

Main Tool: Matrix factorizations.

Def. A matrix factorization of f is a pair of $n \times n$ matrices (φ, ψ) with entries in S s.t.

$$\varphi \psi = f \cdot I_n = \psi \varphi .$$

$MF(f)$ = category of MFs of f.

Ex| $f = x^3 + y^4$

$$\overset{\varphi}{\begin{pmatrix} x & -y^2 \\ y^2 & x^2 \end{pmatrix}} \cdot \overset{\psi}{\begin{pmatrix} x^2 & y^2 \\ -y^2 & x \end{pmatrix}}$$

$$= \begin{pmatrix} x^3 + y^4 & 0 \\ 0 & x^3 + y^4 \end{pmatrix}$$

is a 2×2 MF of $x^3 + y^4$.

Open Question: Given a polynomial f, what is the smallest possible size for a MF of f?

Very few cases are known!

Connection with MCM modules.

Two observations

(1) Given $(\varphi, \psi) \in MF(f)$, both φ and ψ are injective.

$$0 \longrightarrow S^n \xrightarrow{\varphi} S^n \longrightarrow \operatorname{Cok}\varphi \longrightarrow 0$$

(2) $$f(S^n) = \varphi\psi(S^n) \subseteq \operatorname{Im}\varphi$$

$\Rightarrow$ $\operatorname{Cok}\varphi$ is an R-module with

$\operatorname{pd}_S \operatorname{Cok}\varphi = 1$

$\Rightarrow$ $\operatorname{Cok}\varphi \in MCM(R)$ (same for $\operatorname{Cok}\psi$)

Conversely: Let $M \in MCM(R)$

$$\begin{tikzcd}
0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[d, "f"] & S^n \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\psi"'] & M \arrow[r] \arrow[d, "f=0"] & 0 \\
0 \arrow[r] & S^n \arrow[r, "\varphi"'] & S^n \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $(\varphi, \psi) \in MF(f)$ with

$\operatorname{Cok}\varphi = M.$

Theorem (Eisenbud 80')

$$\begin{aligned} MF(f) &\longrightarrow MCM(R) \\ (\varphi,\psi) &\longmapsto \operatorname{Cok}\varphi \end{aligned}$$

induces an equivalence of categories

$$MF(f)/_{\sim} \simeq \underline{MCM}(R)$$

So// Knörrer's Thm can be viewed as:

$$\begin{array}{ccc} \text{R has finite CM type} & \Longleftrightarrow & R^{\#} \text{ has finite CM type} \\ \Big\Updownarrow \text{Eis} & & \Big\Updownarrow \text{Eis} \\ \text{f has finite "MF type"} & \Longleftrightarrow & f+z^2 \text{ has finite "MF type"} \end{array}$$

Ex| $R = \mathbb{C}[\![x,y]\!]/(x^2y)$ is <u>not</u> of finite CM type.

[BGS 87'] For each $k \geq 1$

$$\left(\begin{pmatrix} xy & y^{k+1} \\ 0 & -xy \end{pmatrix}, \begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix} \right)$$

is an indecomposable MF of x^2y.

Knörrer's Theorem:

- $R = S/(f)$
- $R^\# = \dfrac{S[\![z]\!]}{(f+z^2)}$
- $\sigma : R^\# \longrightarrow R^\#$
  $s \longmapsto s$
  $z \longmapsto -z$

$\sigma^2 = 1_{R^\#}$

Form the skew group algebra $R^\#[\sigma]$

- formal sums $a + b\sigma$, $a, b \in R^\#$
- multiplication: $a, b \in R^\#$

$$(a\,\sigma^i)\cdot(b\cdot\sigma^j) = a\,\sigma^i(b)\cdot\sigma^{i+j}.$$

Thm (Knörrer 87')

$$MF(f) \simeq MCM(R^\#[\sigma])$$

where

$$MCM(R^\#[\sigma]) = MCM\, R^\# \cap \operatorname{mod} R^\#[\sigma]$$

$$(R^\# \hookrightarrow R^\#[\sigma])$$

$$
\begin{tikzcd}
R \text{ finite CM type} \arrow[rr, dashed, leftrightarrow] \arrow[d, leftrightarrow, "\text{Eis}"] & & R^{\#} \text{ finite CM type} \arrow[d, leftrightarrow, "\text{Reiten Reidtmann}"] \\
MF(f) \text{ finite type} \arrow[rr, leftrightarrow, "\text{Knö}"] & & R^{\#}[\sigma] \text{ finite type}
\end{tikzcd}
$$

Now consider $d > 2$.

<u>Def</u>. A matrix factorization of f with $d \geq 2$ factors is a tuple of $n \times n$ matrices with entries in S s.t.

$$\varphi_1 \varphi_2 \cdots \varphi_d = f \cdot I_n.$$

Note: $\varphi_i \varphi_{i+1} \cdots \varphi_d \varphi_1 \cdots \varphi_{i-1} = f \cdot I_n \ \forall i$

$MF^d(f)$ = category of d-fold MF of f.

Examples.

(0) $d=3$, For any f

$$(f,1,1),\ (1,f,1),\ (1,1,f)$$

are 1×1 3-fold MFs of f

(1) $f=xyz$, $(x,y,z)\in MF^3(xyz)$

(2) $f=x^3+y^4\in\mathbb{C}[x,y]$ and $\omega\in\mathbb{C}$ is a primitive 3^{rd} root of 1.

$$\begin{pmatrix} y^2 & 0 & x \\ x & y & 0 \\ 0 & x & y \end{pmatrix},\ \begin{pmatrix} y & 0 & \omega x \\ \omega x & y^2 & 0 \\ 0 & \omega x & y \end{pmatrix},\ \begin{pmatrix} y & 0 & \omega^2 x \\ \omega^2 x & y & 0 \\ 0 & \omega^2 x & y^2 \end{pmatrix}$$

is a 3×3 3-fold MF of x^3+y^4.

<u>Rmk.</u> The category $MF^d(f)$ is Frobenius exact for any $d \geq 2$

↳ the indecomposable projective/injective objects are the MFs from example (0)

↳ $\underline{MF}^d(f)$ is triangulated

↳ $\Omega X \cong \Omega^{-1} X$ for all $X \in MF^d(f)$

↳ $\Omega^2 X \cong X$ up to projective summands.

↳ Every object in $MF^d(f)$ has a 2-periodic projective resolution.

Thm (-) $d \geq 2$, $\omega \in S$ a primitive d^{th} root of 1

- $R = S/(f)$
- $R^\# = \dfrac{S[\![z]\!]}{(f + z^d)}$
- $\sigma : R^\# \longrightarrow R^\#$

$$s \longmapsto s$$
$$z \longmapsto \omega z$$

$\sigma^d = 1_{R^\#}$

Form $R^\#[\sigma]$ similar to before.

Then

$$MF^d(f) \simeq MCM(R^\#[\sigma])$$

Where

$$MCM(R^\#[\sigma]) = \left\{ \begin{array}{l} R^\#[\sigma]\text{-modules} \\ MCM \text{ over } R^\# \end{array} \right\}$$

Idea of the proof:

Let $N \in MCM(R^\#[\sigma])$. Then N is MCM over $R^\# \Longrightarrow$ f.g. free over S.

Let $\varphi: N \to N$ be multiplication by z. Pick an S-basis for N and write φ as an $n \times n$ matrix with entries in S.

Then $\varphi^d = \text{mult by } z^d = -f \cdot I_n$

Get a MF of f

$$\approx \underbrace{(\varphi, \varphi, \dots, \varphi)}_{d\text{-times}} \text{ with } d \text{ factors.}$$

Notice that this applies to any MCM $R^\#$-module.

$$\begin{tikzcd} \mathrm{MCM}(R^\#) \arrow[d, shift right] \arrow[drr, bend left, "\flat"] & & \\ \mathrm{MCM}(R^\#[\sigma]) \arrow[u, shift right] \arrow[rr, "\sim"] & & \mathrm{MF}^d(f) \arrow[ull, bend right, "\#"] \end{tikzcd}$$

$\#$ and $\flat$ do not form an equivalence but:

(Leuschke, –) Let $N \in \mathrm{MCM}(R^\#)$ and $X \in \mathrm{MF}^d(f)$.

$$N^{\flat\#} \cong \bigoplus_{i=0}^{d-1} (\sigma^i)^* N \qquad \text{and}$$

$$X^{\#b} \cong \bigoplus_{i=0}^{d-1} T^i(X)$$

where

- $(\sigma^i)^* N$ is the module obtained by restricting scalars along $\sigma^i : R^\# \to R^\#$
- $T^i(\varphi_1, \varphi_2, \ldots, \varphi_d) = (\varphi_i, \varphi_{i+1}, \ldots, \varphi_{i-1})$

Theorem (Leuschke, –)

f has finite d-MF type $\iff$ $R^\# = \dfrac{S[\![z]\!]}{(f+z^d)}$ has finite CM type.

Connecting back to MCM(R)

For $X = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in MF^d(f)$

$X \longmapsto$ chain of surjective maps between MCM R-modules.

$$\operatorname{Cok} \varphi_1\varphi_2\cdots\varphi_{d-1} \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{Cok} \varphi_1\varphi_2 \twoheadrightarrow \operatorname{Cok} \varphi_1 .$$

Thm (Hopkins $d=3$, —)

$$MF^d(f)/\sim \;\; \simeq \;\; \text{epimorphism category of MCM } R$$

$$
\begin{tikzcd}
\text{epimorphism category of } \mathrm{MCM}(R) \arrow[r, leftrightarrow, dashed] \arrow[d, leftrightarrow] & \mathrm{MCM}(R^{\#}) \arrow[d, leftrightarrow] \\
\mathrm{MF}^{d}(f) \arrow[r, leftrightarrow, "\sim"] & \mathrm{MCM}(R^{\#}[\sigma])
\end{tikzcd}
$$