Utah/URiCA talk

Thursday, April 11, 2024 9:35 AM

Matrix Factorizations and Knörrer's Theorem

Joint with Leuschke

Let R be a local ring. A finitely generated module M is called **maximal Cohen Macaulay (MCM)** if

$$\operatorname{depth} M = \dim R.$$

R is called **Cohen-Macaulay (CM)** if the R-module R is MCM.

Why MCM modules? One reason:

$$\underbrace{\text{Structure of } \operatorname{MCM}(R)}_{\text{collection/subcategory of all MCM } R\text{-mods}} \Longrightarrow \text{Structure of } R.$$

Two examples of this idea:

- R is RLR $\Longleftrightarrow$ every MCM is free.
- **Def.** R is said to have **finite CM type** if there are only finitely many indecomposable MCM R-mods up to isomorphism.

Thm (Auslander (complete) '86, Huneke-Leuschke '02)

If R is a CM local ring with finite CM type, then R has at most an isolated singularity.

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$R_{\mathfrak{p}}$ is regular $\forall\, \mathfrak{p} \neq \mathfrak{m}$.

Upshot:

finite-ness property of $MCM(R)$ $\Longrightarrow$ strong Structural constraint on R

fCMt will be the focus for the rest of the talk.

What is Known?

dim $R=0$: finite CM type $\iff$ P.I.R.

eg $k[x]/(x^n)$

dim $R = 1$: Drozd-Roiter give $\iff$ conditions for finite CM type.

Ex| $\dim R = 1$ + finite CM type $\Rightarrow$ $e(R) \leq 3$.

But its not that simple either

$K[[t^3, t^5]]$ ✓ $\qquad$ $K[[t^3, t^7]]$ ✗

dim $R = 2$: If $R = \mathbb{C}[[x,y]]^G$ for a finite group G, then R has fCMt [Herzog '78]

$\dim R > 2$ very little is known except one outlier case:

Hypersurface rings (which are understood in <u>all</u> dimensions)

Thm (Buchweitz - Greuel - Schreyer, Knörrer '87]

Let $S = \mathbb{C}[\![x, y, z_1, \dots, z_r]\!]$, $0 \neq f \in (x, y, z_1, \dots, z_r)^2$, and set $R = S/(f)$. TFAE:

(i) R has finite CM type.

(ii) $R \cong S/(g)$ where g is one of the following

"simple hypersurface singularities"

$$\begin{array}{lll}
(A_n) & x^2 + y^{n+1} + z_1^2 + \cdots + z_r^2 & n \geq 1 \\
(D_n) & x^2 y + y^{n-1} + z_1^2 + \cdots + z_r^2 & n \geq 4 \\
(E_6) & x^3 + y^4 + z_1^2 + \cdots + z_r^2 & \\
(E_7) & x^3 + xy^3 + z_1^2 + \cdots + z_r^2 & \\
(E_8) & \underbrace{x^3 + y^5}_{\text{"dim 1"}} + \underbrace{z_1^2 + \cdots + z_r^2}_{\text{Knörrer's Thm.}} &
\end{array}$$

We will focus on Knörrer's contribution. $\rightsquigarrow$ Matrix Factorizations

Rest of talk: Let S be a regular local ring

$f \in S$ nzd

$R = S/(f)$

Def. A *matrix factorization of f* is a pair (φ, ψ) of matrices over S s.t.

$$\varphi \psi = f \cdot I_n = \psi \varphi.$$

$MF(f)$ = category of all MFs of f.

Ex| $f = x^2$, $\underset{1\times 1}{(x,x)} \in MF(x^2)$

Ex| $f = x^3 + y^4$

$$\underbrace{\begin{bmatrix} y^2 & x \\ x^2 & -y^2 \end{bmatrix}}_{\varphi} \cdot \underbrace{\begin{bmatrix} y^2 & x \\ x^2 & -y^2 \end{bmatrix}}_{\psi} = \begin{bmatrix} x^3+y^4 & 0 \\ 0 & x^3+y^4 \end{bmatrix} \underset{\text{also}}{= \psi\varphi}$$

So, $(\varphi, \psi) \in MF(x^3+y^4)$.

Connection to MCM $R = S/(f)$ - modules.

(1) $\varphi\psi = f \cdot I_n = \psi\varphi \Rightarrow \varphi, \psi$ are both injective since f is nzd.

$\Rightarrow$ Have $0 \to S^n \xrightarrow{\varphi} S^n \longrightarrow \operatorname{Cok}\varphi \to 0$

$\Rightarrow$ $\operatorname{pd}_S \operatorname{Cok}\varphi = 1$

(2) If $x \in S^n$, $f \cdot x = \varphi\psi(x) \in \operatorname{Im}\varphi$

$$\Rightarrow f \cdot \operatorname{Cok}\varphi = 0$$

So $M = \operatorname{Cok}\varphi$ is an $R = S/(f)$ - module!

By AB Formula, $\operatorname{depth} M = \dim S - \operatorname{pd}_S M$

$$= \dim S - 1$$

$$= \dim S - 1$$
$$= \dim R$$

$\Rightarrow$ $\operatorname{Cok}\varphi$ is an MCM R-module (same for $\operatorname{cok}\psi$)

Conversely: Let $M \in MCM(R)$. $\qquad S \twoheadrightarrow R = S/(f)$

Again by AB $\Rightarrow$ $\operatorname{pd}_S M = 1$.

(localize at f to see $n = m$)

$$\begin{tikzcd}
① \ \exists & 0 \arrow[r] & S^{\cancel{m}\, n} \arrow[r, "\varphi"] \arrow[d, "f"'] & S^n \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\psi"'] & M \arrow[r] \arrow[d, "f = 0"] & 0 \\
② & 0 \arrow[r] & S^n \arrow[r, "\varphi"'] & S^n \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

M is an R-mod so $f = 0$

So, mult by f and 0 lift the same map on M

$\Rightarrow$ f is homotopic to 0

$\Rightarrow \exists\ \psi : S^n \longrightarrow S^n$ s.t. $\varphi\psi = f \cdot I_n$ $\quad$ ($\psi\varphi = f \cdot I_n$ follows)

So, $(\varphi, \psi) \in MF(f)$ and has $\operatorname{Cok}\varphi = M$.

Thm (Eisenbud '80)

$$MF(f) \longrightarrow MCM(R)$$
$$(\varphi, \psi) \longmapsto \operatorname{Cok}\varphi$$

induces a bijection between MFs of $f/\sim$

induces a bijection between MFs of $f/\sim$
and MCM R-mods.

(reduced MFs $\Longleftrightarrow$ stable MCMs)

Consequence: Convert our question about fCMt to linear algebra!

$R = S/(f)$ has finite CM type $\iff$ f has finite MF type

The linear algebra really helps!

$\boxed{\text{Ex}}$ (BGS '87)

$R = \dfrac{K[\![x,y]\!]}{(x^2)}$ $\qquad \left(\begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix}, \begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix} \right) \in MF(x^2)$

$f = x^2$ $\qquad$ is an indecomposable MF $\forall\, n \geq 1$

No longer an isolated singularity. $\qquad \Rightarrow$ R does not have fCMt.

non-iso for $n \neq m$

* one other takeaway from this example: it really matters which ring we view our polynomial in.

$MF_S(f)$

Now Knörrer's Theorem.

Thm (Knörrer '87) Let (S, n, K) be a complete RLR with $K = \overline{K}$ and char $K \neq 2$.

$$MF_S(f) \xleftrightarrow{\text{Knörrer}} MF_{S[\![z]\!]}(f + z^2)$$

$$\begin{array}{ccc}
\begin{array}{c} MF_S(f) \\ f \text{ has finite} \\ MF \text{ type} \end{array} & \overset{\text{Knorrer}}{\Longleftrightarrow} & \begin{array}{c} MF_{S[[z]]}(f+z^2) \\ f+z^2 \text{ has finite} \\ MF \text{ type.} \end{array} \\
\Big\Updownarrow \text{Eisenbud} & & \Big\Updownarrow \text{Eisenbud} \\
\begin{array}{c} R = S/(f) \\ \text{has finite} \\ CM \text{ type} \end{array} & \underset{\text{Knörrer}}{\Longleftrightarrow} & \begin{array}{c} R^\# \overset{def}{=} \dfrac{S[[z]]}{(f+z^2)} \\ \text{has finite CM} \\ \text{type} \end{array}
\end{array}$$

"double branched cover of R"

Ex| A_n singularity of dim r has fCMt $\forall\, n \geq 1$ and $r \geq 0$

$$\underset{\text{fCMt}}{y^{n+1}} \Rightarrow \underset{\text{fCMt}}{y^{n+1} + x^2} \Rightarrow \underset{\text{fCMt}}{y^{n+1} + x^2 + z_1^2} \Rightarrow \cdots \Rightarrow \begin{array}{c} A_n \text{ of} \\ \text{dim } r \\ \text{fCMt} \end{array}$$

How does Knörrer use MFs?

Given $(\underbrace{\varphi, \psi}_{n \times n}) \in MF_S(f)$, define

$$\underset{f}{(\varphi, \psi)} \otimes \underset{z^2}{(z, z)} \overset{def}{=} \left(\begin{pmatrix} z & \varphi \\ \psi & -z \end{pmatrix}, \begin{pmatrix} z & \varphi \\ \psi & -z \end{pmatrix} \right) \in MF_{S[[z]]}(\underline{\underline{f+z^2}})$$

$z = z \cdot I_n$

Get functors

$$MF(f) \underset{\text{mod } z \,=\, \overline{(-)}}{\overset{-\otimes(z,z)}{\rightleftarrows}} MF(f+z^2)$$

These almost compose to id:

(1) $\overline{(\varphi,\psi)\otimes(z,z)} = \overline{\left(\begin{pmatrix} z & \varphi \\ \psi & -z \end{pmatrix}, \begin{pmatrix} z & \varphi \\ \psi & -z \end{pmatrix}\right)}$

$(\varphi,\psi)\in MF_S(f)$

$$= \left(\begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}\right)$$

$$\cong (\varphi,\psi) \oplus (\psi,\varphi)$$

So// $\underset{\text{fCMt}}{R^\#} \Longrightarrow \underset{\text{fCMt}}{R}$

(2) Harder direction: If $(\Phi,\Psi)\in MF_{S[[z]]}(f+z^2)$

$$\overline{(\Phi,\Psi)}\otimes(z,z) = (\Phi,\Psi)\oplus(\Psi,\Phi)$$

So// $\underset{\text{fCMt}}{R} \Rightarrow \underset{\text{fCMt}}{R^\#}$

d-fold Matrix Factorizations

A d-fold matrix factiorization of f is $(\varphi_1,\varphi_2,\dots,\varphi_d)$

s.t. $\varphi_1\varphi_2\cdots\varphi_d = f\cdot I_n$

$\varphi_2\varphi_3\cdots\varphi_d\varphi_1 = f\cdot I_n$

$\vdots$

$MF^d(f)$ = category of d-fold MFs of f.

Ex $(x, x, x) \in MF^3(x^3)$ is a 3-fold MF

Ex If $(\varphi_1, \varphi_2, \varphi_3) \in MF^3(f)$, then

$$\underbrace{(\varphi_1, \varphi_2, \varphi_3)}_{f} \otimes \underbrace{(z, z, z)}_{z^3} \in MF^3(\underline{f + z^3})$$

Let
$$\varphi = \begin{pmatrix} z & \varphi_1 & 0 \\ 0 & \omega z & \varphi_2 \\ \varphi_3 & 0 & \omega^2 z \end{pmatrix}.$$
Then $\varphi^3 = (f + z^3) \cdot I_3$

$\Rightarrow (\varphi, \varphi, \varphi) \in MF(f + z^3)$

$z = z \cdot I_n$, $\omega^3 = 1$

Application (Backelin-Herzog-Ulrich '91)

Every hypersurface ring has an Ulrich module

$\mu(M) = e(M)$

- Easy to construct using $\otimes$!
- [Iyengar-Ma-Walker-Zhuang] recently showed $\exists$ 2 dim'l complete intersections w/o any Ulrich modules.

Q: Which f have only finitely many indecomposable d-fold MFs?

Thm (Leuschke-T) Let $S = \mathbb{C}[[x, z_1, \dots, z_r]]$ and $f \in (x, z_1, \dots, z_r)^2$. If $d > 2$, the f has finite dMF type iff f and d are one of the following:

(A_1) $x^2 + z_1^2 + \cdots + z_r^2$ $\quad d > 2$

(A_2) $x^3 + z_1^2 + \cdots + z_r^2$ $\quad d = 3, 4, 5$

(A_1) $\quad x^2 + z_1^2 + \cdots + z_r^2 \qquad d > 2$

(A_2) $\quad x^3 + z_1^2 + \cdots + z_r^2 \qquad d = 3, 4, 5$

(A_3) $\quad x^4 + z_1^2 + \cdots + z_r^2 \qquad d = 3$

(A_4) $\quad x^5 + z_1^2 + \cdots + z_r^2 \qquad d = 3$

How?

$MF^d(f)$ has finite type $\iff$ $R^\# = \dfrac{S[\![z]\!]}{(f + z^d)}$ has fCMt